MATH 425 11/14/2011

The χ^2 distribution with k degrees of freedom
= sum of k independent random variables which are squares of standard Gaussian variables.

The task: Write down the probability density function.

$k=1$: $X = Y^2$ where Y is a standard Gaussian.

Probability density function of Y

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}.$$

Probability density for X is related by a change of variable

$$x = y^2 \qquad\qquad dx = 2y\,dy$$

$$y = \sqrt{|x|} \qquad\qquad dx = 2\sqrt{x}\,dy \qquad\qquad (x \geq 0)$$

$$dy = \frac{dx}{2\sqrt{x}}$$

$$\frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy = \frac{1}{\sqrt{2\pi}} e^{-x/2} \cdot \frac{dx}{2\sqrt{x}}$$

But we have two values of y to each value of x (except $x = 0$ - has measure 0).

So we must multiply by 2:

$$\boxed{\frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/2} = g(x), \quad x \geq 0 \qquad g(x) = 0 \quad x < 0.}$$

$g(x)$ = probability density function for χ^2 with one degree of freedom.

<u>For general k</u>: We have a sum of k independent variables each of which has the distribution χ^2 with $k=1$.

$$x_1 + \cdots + x_k = x$$

(k-1)-fold integral

$$\int_{x_1+\cdots+x_k=x} \cdots \iint g(x_1)\cdot \ldots \cdot g(x_k) = \boxed{\int_{x_1+\cdots+x_k=x} \cdots \int (x_1 \cdots x_k)^{-1/2}}\, e^{-x/2}$$

how does this depend on x?

$$u_1 = \frac{x_1}{x}, \quad \ldots \quad u_k = \frac{x_k}{x} \qquad u_1 + \cdots + u_k = 1$$

$$du_1 = \frac{dx_1}{x}, \ldots, du_k = \frac{dx_k}{x}$$

$$\left.dx_1 = x\,du_1, \quad \ldots \quad dx_k = x\,du_k \right\}$$ we only need $(k-1)$ of them.

So the substitution gives:

$$x^{k-1} \boxed{\int_{u_1 + \cdots + u_k = 1} \cdots \int (u_1 \cdots u_k)^{-1/2} x^{-k/2}}$$

$x_i = x u_i$

$$= x^{(k-1)-k/2} \boxed{\int_{u_1 + \cdots + u_k = 1} \cdots \int (u_1 \cdots u_k)^{-1/2}}$$

does not depend on x.

The χ^2 distribution with k degrees of freedom is

$$\begin{cases} \boxed{C\, x^{\frac{k}{2}-1} e^{-x/2}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

What is the constant?

$$C = \frac{1}{\int_0^\infty x^{\frac{k}{2}-1} e^{-x/2}\, dx}$$

<u>The standard notation</u>: The Gamma (Γ-) function.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad z > 0.$$

The Gamma function is not an elementary function. <u>But</u>, doing integration by parts,

$u = t^{z-1}$ $\qquad v' = e^{-t}$

$u' = (z-1)t^{z-2}$ $\qquad v = -e^{-t}$

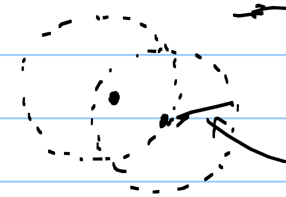
$$\boxed{\Gamma(z) = (z-1)\,\Gamma(z-1)}$$

$$= (z-1)(z-2)\,\Gamma(z-2) =$$

$$= (z-1)(z-2)(z-3)\,\Gamma(z-3)$$

(Note: complex analysis lets us extend this function to all complex numbers z except $z = 0, -1, -2, \cdots$.)

<u>Uses Taylor series</u>



radius of convergence

$$\Gamma(0) = \int_0^\infty e^{-t}\,dt = -e^{-t}\Big|_0^\infty = 1$$

If n is a positive integer,

$$\boxed{\Gamma(n) = (n-1)!}$$

Now back to calculating the constant C as above:

$$\frac{1}{C} = \int_0^\infty x^{\frac{k}{2}-1} e^{-x/2}\,dx = \qquad \text{Put } t = \frac{x}{2}$$

$$x = 2t \qquad dx = 2\,dt$$

$$= 2^{\frac{k}{2}} \int_0^\infty t^{\frac{k}{2}-1} e^{-t}\,dt = 2^{\frac{k}{2}}\,\Gamma\left(\frac{k}{2}\right).$$

The χ^2 distribution has probability density function

$$\boxed{\begin{cases} \dfrac{1}{2^{\frac{k}{2}}\Gamma\left(\frac{k}{2}\right)} x^{\frac{k}{2}-1} e^{-x/2} & x \geq 0 \\ 0 & x < 0. \end{cases}}$$

Back to $k=1$: $\dfrac{1}{\sqrt{2\pi}} x^{-\frac{1}{2}} e^{-x/2}$

$$2^{\frac{1}{2}}\Gamma\left(\tfrac{1}{2}\right) = \sqrt{2\pi} \qquad \boxed{\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}}$$

HW ① Derive a formula for $\Gamma\left(k+\frac{1}{2}\right)$, $k \in \mathbb{Z}$.

(2) Compute the hazard rate function for the χ^2-distribution with k degrees of freedom.