MATH 425 11/18/2011

Note Title 11/18/2011

<u>The Student t distribution with k degrees of freedom</u>

$$t = \frac{X}{\sqrt{\frac{Y}{k}}}$$

X = standard Gaussian

Y = χ^2 with k degrees of freedom.

<u>Goal</u>: the probability density function.

Ignore $\sqrt{k}$ for now.

Let us first look at the probability density of $\sqrt{Y}$.

Let us ignore the multiplicative constant of probability density.

Probability density of Y:

$$C\, y^{\frac{k}{2}-1} e^{-y/2}\, dy$$

Let $U = \sqrt{Y}$. Then $u = \sqrt{y}$.

$y = u^2$

$dy = 2u\,du$

Substitution:

$$y^{\frac{k}{2}-1} e^{-y/2}\, dy = \text{const.}\, u^{k-2} e^{-u^2/2}\, u\, du =$$

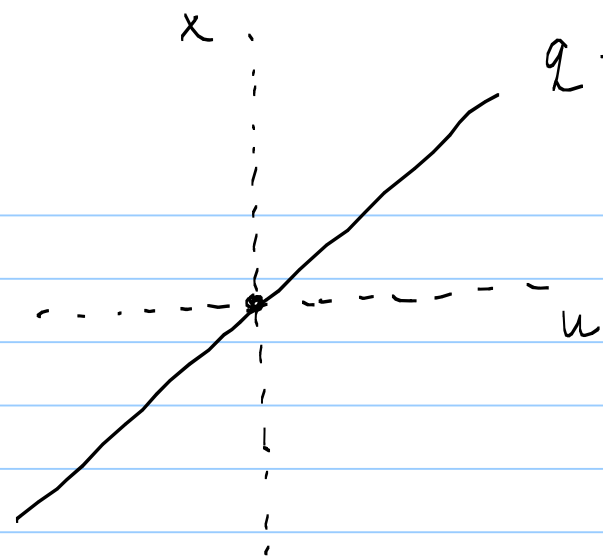
$$= \text{const.}\, u^{k-1} e^{-u^2/2}\, du$$

Probability density function of U: const. $u^{k-1} e^{-u^2/2}$.

let us look at

$$q = \frac{X}{U}$$

$$t = \sqrt{k} \cdot q$$



$q = \frac{x}{u} = \text{constant}.$

line integral according to arc length (Calculus 3)

$$dq\,du = \left| \det \begin{pmatrix} \partial q/\partial x & \partial q/\partial u \\ \partial u/\partial x & \partial u/\partial u \end{pmatrix} \right| dx\,du$$

$$= \left| \det \begin{pmatrix} \frac{1}{u} & -\frac{x}{u^2} \\ 0 & 1 \end{pmatrix} \right| dx\,du = \frac{1}{u}\,dx\,du$$

$$\therefore\ dx\,du = u\,dq\,du$$

$$\text{Const} \int_{-\infty}^{\infty} u^{k-1} e^{-u^2/2} e^{-q^2u^2/2} \, u \quad du =$$

$$= \text{const} \int_{-\infty}^{\infty} u^k \quad e^{-(1+q^2)u^2/2} \, du =$$

$$v = \sqrt{1+q^2} \; u$$

$$u = \frac{v}{(1+q^2)^{1/2}}$$

$$du = \frac{1}{\sqrt{1+q^2}} \, dv$$

$$= \text{const.} \int_{-\infty}^{\infty} \frac{v^k}{(1+q^2)^{k/2}} e^{-v^2/2} \frac{1}{\sqrt{1+q^2}} \, dv$$

$$= \text{const.} \frac{1}{(1+q^2)^{(k+1)/2}} \int_{-\infty}^{\infty} v^k e^{-v^2/2} \, dv \Leftarrow \text{const.}$$

$$\sim \frac{1}{(1+q^2)^{(k+1)/2}}$$

Note: We can actually do the integral

$$\int \frac{1}{(1+q^2)^{(k+1)/2}}\,dq$$

To calculate the scaling constant.

At this point, let us recall that

$$t = \sqrt{k}\, q$$

So the $\boxed{\text{probability density of } t}$ is !!!

$$\text{const.}\left(1+\frac{t^2}{k}\right)^{-\frac{k+1}{2}} = \boxed{\frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma\left(\frac{k}{2}\right)}\left(1+\frac{t^2}{k}\right)^{-\frac{k+1}{2}}}$$

Example: When $k = 1$

$$\text{const.}\ (1+t^2)^{-1} = \text{const} \cdot \frac{1}{1+t^2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}\, dt = \arctan t \Big|_{-\infty}^{\infty} = \pi$$

This probability density is $\boxed{\frac{1}{\pi} \cdot \frac{1}{1+t^2}}$

The Cauchy distribution

The limit of t as $k \to \infty$ is the standard Gaussian.

$$\lim_{k\to\infty} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma\left(\frac{k}{2}\right)} \left(1+\frac{t^2}{k}\right)^{-\frac{k+1}{2}} =$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{k\pi}} \cdot \frac{\cancel{\sqrt{2\pi\frac{k-1}{2}}}}{\cancel{\sqrt{2\pi\frac{k-2}{2}}}} \frac{\left(\frac{k-1}{2e}\right)^{\frac{k-1}{2}}}{\left(\frac{k-2}{2e}\right)^{\frac{k-2}{2}}} \left(1+\frac{t^2}{k}\right)^{-\frac{k+1}{2}} =$$

<u>Stirling formula</u>:

$$\lim_{k\to\infty} \frac{\Gamma(k+1)}{\sqrt{2\pi k}\left(\frac{k}{e}\right)^k} = 1.$$

$$= \lim_{k\to\infty} \frac{1}{\sqrt{\cancel{k}\pi}} \frac{1}{\sqrt{2e}} \underbrace{\left(1+\frac{1}{k-2}\right)^{\frac{k-2}{2}}}_{\sqrt{e}} \cdot \cancel{\sqrt{k-1}} \cdot \underbrace{\left(1+\frac{t^2}{k}\right)^{-\frac{k+1}{2}}}_{e^{-t^2/2}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \leftarrow \text{\underline{standard Gaussian}}$$

$$\boxed{\lim_{k\to\infty} \left(1+\frac{x}{k}\right)^k = e^x}$$

HW: 1 Express $\dfrac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\,\Gamma\left(\frac{k}{2}\right)}$ in terms of factorials and π (Hint: do k even and odd separately)

(2) Explain how to do the integrals

(a) $\displaystyle\int \frac{1}{(1+t^2)^k}\,dt \qquad k = 1, 2, \ldots$

(b) $\displaystyle\int \frac{1}{(1+t^2)^{k+\frac{1}{2}}}\,dt \qquad k = 1, 2, \ldots$

Hint: (i) $\displaystyle\frac{1}{(1+t^2)^{\ell-1}} = \frac{1+t^2}{(1+t^2)^{\ell}} = \frac{1}{(1+t^2)^{\ell}} + \frac{t^2}{(1+t^2)^{\ell}}$

(iii)

$$u = \frac{1}{(1+t^2)^{\ell-1}} \qquad v' = 1$$

$$u' = (1-\ell)\frac{2t}{(1+t^2)^{\ell}} \qquad v = t$$

$$u'v = 2(1-\ell)\frac{t^2}{(1+t^2)^{\ell}}$$