MATH 425 11/21/2011

Note Title 11/21/2011

The Γ-function has interesting applications.

Crazy question: What is the $\frac{1}{2}$'th derivative of a function? (??)

put any number here.

One can actually make such definitions with the Γ-function.

<u>Put</u>

$$D^{-1} f(t) = \int_0^t f(x)\, dx \quad \Big\}\ \text{"the antiderivative"}$$

It turns out that we have

$$D^{-n} f(t) = \frac{1}{(n-1)!} \int_0^t (t-x)^{n-1} f(x)\, dx$$

$$\frac{d D^{-n} f(t)}{dt} = \frac{1}{(n-1)!} \int_0^t (n-1)(t-x)^{n-2} f(x)\, dx$$

$$= \frac{1}{(n-2)!} \int_0^t (t-x)^{n-2} f(x)\, dx = D^{-(n-1)} f(t).$$

(This works for $n = 1, 2, 3, \ldots$.)

Define $$D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-x)^{\nu-1} f(x)\, dx.$$

$$\underbrace{\phantom{D^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-x)^{\nu-1} f(x)\, dx}}_{\nu \in \mathbb{C} \setminus \{0, -1, -2, \dots\}}$$

for these, you can use f and its ordinary derivatives.

Filling in some theory

The probability density of a function of a random variable.

Let u be a function of a random variable X.
$u = u(x)$.

Let x have a density f. What is the density of u? What we already used: when transforming a density, include dx:

X .... $f(x)\,dx$

convert to du

U ..... $f(x(u))\,dx = f(x(u))\left(\frac{dx}{du}\right)du$

$$du = \frac{du}{dx}\,dx$$

$$dx = \left(\frac{du}{dx}\right)^{-1} du = \left(\frac{dx}{du}\right) du$$

The density of U is

$$f(x(u))\,\frac{dx(u)}{du}. \qquad (*)$$

If we introduce a separate name, say, g, for the function expressing u in terms of x.

$$u = g(x).$$

Then we want to write

$$x = g^{-1}(u).$$

We should assume that g is <u>monotone</u> (increasing or decreasing).

add an absolute value to the integrating factor.

The formula (*) becomes

Density of $U = g(X)$.

$$\begin{cases} f(g^{-1}(u)) \left| \dfrac{d\, g^{-1}(u)}{du} \right| & \text{if } u = g(x) \text{ for some } x \\ & (g^{-1}(u) \text{ is defined}) \\ 0 & \text{else.} \end{cases}$$

Example: (7d on p. 221) Let X be a continuous **non-negative** random variable with density f, let $Y = X^n$. Find the probability density of Y.

Solution: $g(x) = x^n$

$$\frac{d\left(y^{1/n}\right)}{dy} = \frac{1}{n}\, y^{(1-n)/n}$$

Answer:

$$\frac{f(y^{1/n})}{n} \; y^{(1-n)/n}.$$

Jointly distributed random variables

On the same sample spaceS.

$X : S \longrightarrow \mathbb{R}$

$Y : S \longrightarrow \mathbb{R}$

Joint cummulative distribution $a, b \in \mathbb{R}$

$$F(a,b) = P\{X \le a, \; Y \le b\}$$
$$= P(\{X \le a\} \cap \{Y \le b\})$$

Marginal distributions - the individual cummulative distributions of X, Y.

If X, Y are discrete, there is a joint probability mass function

$$p(x,y) = P\{X = x, Y = y\} = P(\{X = x\} \cap \{Y = y\})$$

Then we can write the joint cummulative distribution as

$$F(a,b) = \sum_{\substack{x \le a \\ y \le b}} p(x,y).$$

only countably many non-zero summands.

We say that X, Y are jointly (absolutely) continuous if for every measurable set $C \subseteq \mathbb{R}^2$ we have

$$P\{(X,Y) \in C\} = \iint_C f(x,y)\,dx\,dy. \qquad (*)$$

(the Lebesgue integral).

This is actually equivalent to assuming this for the special case of the joint cummulative distribution $C = (-\infty, a] \times (-\infty, b]$
$(*)$ becomes:

$$F(a,b) = \int_{x=-\infty}^{a} \int_{y=-\infty}^{b} f(x,y)\,dx\,dy$$

We see that $f(a,b) = \dfrac{\partial^2 F(a,b)}{\partial a\, \partial b}$ almost everywhere.

$$\boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1}$$

Example: Let the joint density be

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0 < x < \infty,\ 0 < y < \infty \\ 0 & \text{else.} \end{cases}$$

Compute $P\{X < Y\}$.

$$\int_{\{x \le y\}} 2e^{-x}e^{-2y}\,dx\,dy = \text{Fubini Theorem}$$

$$= \int_{y=0}^{\infty} \int_{x=0}^{y} 2e^{-x} e^{-2y} \, dx \, dy =$$

$$= \int_{y=0}^{\infty} -2e^{-x} e^{-2y} \Big|_{x=0}^{y} dy = \int_{y=0}^{\infty} \left(2e^{-2y} - 2e^{-3y}\right) dy$$

$$= -e^{-2y} + \frac{2}{3} e^{-3y} \Big|_{0}^{\infty} = 1 - \frac{2}{3} = \underline{\underline{\frac{1}{3}}}$$

HW ① 5.29 p. 228

② 5.30 p. 228

(3) Let $f(x,y) = \begin{cases} 2e^{-x}e^{-2y} & 0<x<\infty,\ 0<y<\infty \\ 0 & \text{else.} \end{cases}$

Compute $P\{X < kY\}$ $\quad k \geq 0$ a constant.

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