MATH 425 11/23/2011

Note Title 11/23/2011

$$\int \frac{1}{(1+x^2)^k}\,dx$$

$$\frac{1}{(1+x^2)^k} = \frac{1+x^2}{(1+x^2)^{k+1}} = \frac{1}{(1+x^2)^{k+1}} + \frac{x^2}{(1+x^2)^{k+1}}$$

$$u = \frac{1}{(1+x^2)^k} \qquad v' = 1 \qquad \int uv' = uv - \int u'v$$

$$u' = (-2k)\frac{x}{(1+x^2)^{k+1}} \qquad v = x \qquad \frac{x^2}{(1+x^2)^{k+1}} = \frac{1}{(1+x^2)^k} - \frac{1}{(1+x^2)^{k+1}}$$

$$\int \frac{1}{(1+x^2)^k}\,dx = \frac{x}{(1+x^2)^k} + 2k\int \frac{x^2}{(1+x^2)^{k+1}}\,dx =$$

$$= \frac{x}{(1+x^2)^k} + 2k\int \frac{1}{(1+x^2)^k}\,dx - 2k\int \frac{1}{(1+x^2)^{k+1}}\,dx$$

$$\boxed{\int \frac{1}{(1+x^2)^{k+1}}\,dx = \frac{1}{2k}\,\frac{x}{(1+x^2)^k} + \frac{2k-1}{2k}\int \frac{1}{(1+x^2)^k}\,dx}$$

$$\int \frac{1}{\sqrt{1+x^2}}\,dx = \operatorname{arcsinh}(x) \qquad ?$$

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\sinh(x)' = \frac{e^x + e^{-x}}{2} = \cosh(x)$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\sinh(x)' = \sqrt{1 + \sinh(x)^2}$$

$$\operatorname{arcsinh}(t)' = \frac{1}{\sqrt{1+t^2}}$$

inverse function

$$\frac{e^x - e^{-x}}{2} = t$$

$$e^x - e^{-x} = 2t \quad \Big|\, e^x$$

$$(e^x)^2 - 1 = 2t e^x$$

$$(e^x)^2 - 2t(e^x) - 1 = 0$$

$$\underset{>0}{e^x} = \frac{2t + \sqrt{4t^2 + 4}}{2}$$

$$x = \ln \frac{2t + \sqrt{4t^2+4}}{2} = \ln\left(t + \sqrt{t^2+1}\right)$$

$$\boxed{\int \frac{1}{\sqrt{1+t^2}}\,dt = \ln\left(t + \sqrt{t^2+1}\right) + C}$$

<u>k jointly distributed random variables</u>

$X_1, \ldots, X_k$

(1) discrete (each only attains at most countably many values)

Joint probability mass function:

$$p(x_1, \cdots x_k) = P(\{X_1 = x_1\} \cap \cdots \cap \{X_k = x_k\})$$

Cummulative joint distribution.

$$F(x_1, \ldots, x_k) = \sum_{\substack{t_1 \le x_1 \\ \vdots \\ t_k \le x_k}} p(t_1, \ldots, t_k)$$

(absolutely)

2. <u>Continuous</u>: There exist a joint probability density function $f(x_1, \ldots, x_k)$

$$P\{(X_1, \ldots, X_k) \in \underset{\substack{\cap\!\mid \\ S}}{A}\} = \underbrace{\int \cdots \int_A}_{k\text{-fold integral}} f(x_1, \ldots, x_k)\, dx_1 \cdots dx_k .$$

(suffices to verify for $A = (-\infty, x_1) \times \cdots \times (-\infty, x_k)$, which is the case of the joint cummulative distribution:

$$F(x_1, \dots, x_k) = \int_{t_1=-\infty}^{x_1} \cdots \int_{t_k=-\infty}^{x_k} f(x_1, \dots, x_k)\, dx_1 \dots dx_k .$$

Independent random variables

$$P(\{X_1 \in \mathcal{A}_1\} \cap \dots \cap \{X_k \in \mathcal{A}_k\}) =$$

$$= P\{X_1 \in \mathcal{A}_1\} P\{X_2 \in \mathcal{A}_2\} \dots P\{X_k \in \mathcal{A}_k\}$$

If you look at the case where

$\mathcal{A}_1 = \{x_1\}, \dots, \mathcal{A}_k = \{x_k\}$

and $X_1, \dots, X_k$ are discrete then we get for the joint probability mass function

$$p(x_1, \dots, x_k) = \underbrace{p_{X_1}(x_1)} \cdot \ldots \cdot \underbrace{p_{X_k}(x_k)}$$

the individual (marginal) probability mass functions.

In the continuous case, we get for the joint probability density

$$f(x_1, \dots, x_k) = \underbrace{f_{X_1}(x_1) \cdot \ldots \cdot p_{X_k}(x_k)}_{\text{product of the marginal densities}}$$

Example 1: The multinomial distribution.

For $n=1$ (the "Bernoulli" case):

we have a trial with k possible outcomes, outcome i has probability p_i. (k exclusive Bernoulli variables, $X_1, \dots X_k$, $X_1 + \dots + X_k = 1$).

For general n, iterate this n times, independently ($X_{i,j}$, $X_{i',j'}$ independent if $j \neq j'$)

$$\left.\begin{matrix} X_{1,1} & X_{1,2} & \cdots & X_{1,n} \\ \vdots & \vdots & & \vdots \\ X_{k,1} & X_{k,2} & & X_{k,n} \end{matrix}\right\} \begin{matrix} X_1 = X_{1,1} + \dots + X_{1,n} \\ \\ X_k = X_{k,1} + \dots + X_{k,n} \end{matrix}$$

$X_1, \ldots X_k$ are <u>multinomially distributed.</u>

Probability mass function

$$p(n_1, \ldots, n_k) = \binom{n}{n_1, \ldots, n_k} p_1^{n_1} \cdot \ldots \cdot p_k^{n_k} \text{ if } n_1 + \cdots + n_k = n$$

$(\geq 0 \text{ integers})$

$$= 0 \quad \text{else.}$$

<u>A more specific example:</u>

Suppose we roll a die 10 times. What is the probability of getting 2 one's, 3 two's, 0 three's, 5 four's, 0 fives or six's.

$$\binom{10}{2,3,0,5,0,0}\left(\frac{1}{6}\right)^{10} = \binom{10}{2,3,5}\left(\frac{1}{6}\right)^{10} =$$

$$= \frac{10!}{2!\,3!\,5!}\left(\frac{1}{6}\right)^{10}.$$

HW ① 6.24 on p. 288 (skip (d), (e))

② 6.20