MATH 425 11/28/2011

Note Title 11/28/2011

# The multivariable Gaussian

We want a joint probability density in $n$ variables $\vec{x} = (x_1, \dots, x_n)$. What is the analogue of $e^{-x^2/2\sigma^2}$?

What is the analogue of $x^2/\sigma^2$?

$= x^2 a^2 \qquad \sigma = 1/a$

$\vec{x} A \vec{x}^T$

a symmetric $n \times n$ matrix (some other condition generalizing $a^2 > 0$)

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

symmetric means

$a_{ij} = a_{ji}$

Example: $\begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} = A$

$\vec{x} = (x, y, z) \qquad \vec{x}^T = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 4 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 4x + y \\ x + 3y + z \\ y + 2z \end{pmatrix} =$$

$$= 4x^2 + \underline{xy} + \underline{xy} + 3y^2 + \underline{yz} + \underline{zy} + 2z^2$$

A quadratic form (occurs in multivariable calculus) $\rightarrow$ $= 4x^2 + 2xy + 3y^2 + 2yz + 2z^2$

diagonal entries

off-diagonal entries multiplied by 2.

centered $\leftarrow$ The expected value is $\vec{0}$.

The multivariable Gaussian joint density is

$$C\, e^{-\vec{x} A \vec{x}^T / 2}$$

For a general multivariable expected value

$\vec{\mu} = (\mu_1, \dots, \mu_n)$,

$$C\, e^{-(\vec{x} - \vec{\mu}) A (\vec{x} - \vec{\mu})^T / 2}$$

The matrix A must be <u>positive-definite</u> which means that for any vector $\vec{x} = (x_1, \ldots, x_n)$, we have

$$\vec{x} A \vec{x}^T > 0.$$

$$\vec{x}^T = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

(This is the analogue of $a > 0$.)

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It turns out that after a <u>change of variables</u>, this is essentially just $\underline{n}$ independent single-variable Gaussians.

**Example**: setting a quadratic form equal to a constant: $4x^2 + 2xy + 3y^2 + 2yz + 2z^2 = K$

This is a quadratic surface (quadric in $\mathbb{R}^3$).

In $\mathbb{R}^2$, a quadric is called a conic:

$$4x^2 + xy + 4y^2 = K$$

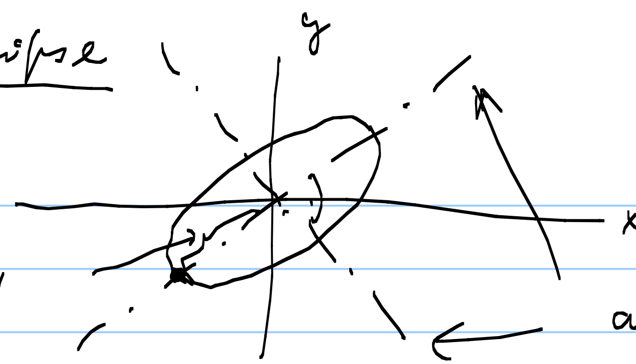
ellipse
hyperbola
parabola

In $\mathbb{R}^2$, we can solve the problem:

$$\underbrace{4x^2}_{(2x)^2} + 2\cdot(2x)\cdot\frac{y}{4} + \left(\frac{y}{4}\right)^2 + \left(4 - \frac{1}{16}\right)y^2 = K$$

$\to \left(2x + \frac{y}{4}\right)^2 + \underbrace{\left(4 - \frac{1}{16}\right)}_{>0 \text{ positive definite}} y^2 = K$

an ellipse




the longest length of a vector which is on the ellipse.

an ellipse has a choice of perpendicular axes!

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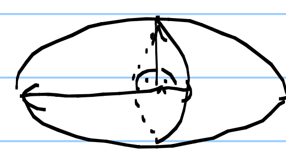
Similarly in 3d: Setting a positive-definite quadratic form equal to a constant $>0$, the set of solutions is an ellipsoid.



there are always 3 perpendicular axes, called main axes

In dim $n$, we get an "$n$-dimensional ellipsoid", again there are $n$ perpendicular main axes.

When we change variables from $\underbrace{x, y, z}^{3d}$ to $u, v, w$, coordinates with respect to the main axes of the quadratic form, we get

$$C\, e^{-(u_1^2 \lambda_1 + \cdots + u_n^2 \lambda_n)/2}, \quad \lambda_1, \ldots, \lambda_n > 0,$$

$$= C\, e^{-u_1^2 \lambda_1/2} \cdot \ldots \cdot e^{-u_n^2 \lambda_n/2}$$

What constant do we have to multiply

$$e^{-u_k^2 \lambda_k/2} \qquad \lambda_k = \frac{1}{\text{variance}} = \frac{1}{\sigma^2}$$

how to integrate to 1 over $\mathbb{R}$?

Answer: $$\frac{\sqrt{\lambda_k}}{\sqrt{2\pi}} = \frac{1}{\sigma\sqrt{2\pi}}. \qquad \sigma = \frac{1}{\sqrt{\lambda_k}}$$

So we find

$$C = \frac{\sqrt{\lambda_1}}{\sqrt{2\pi}} \cdot \ldots \cdot \frac{\sqrt{\lambda_n}}{\sqrt{2\pi}} = \frac{\sqrt{\lambda_1 \cdot \ldots \cdot \lambda_n}}{(2\pi)^{n/2}} = \frac{\sqrt{\det A}}{(2\pi)^{n/2}}.$$

$$\boxed{\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc} \qquad \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$= a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Summary: The general $n$-variable Gaussian has joint probability density

$$\frac{\sqrt{\det A}}{(2\pi)^{n/2}} \, e^{-(\vec{x}-\vec{\mu})\, A\, (\vec{x}-\vec{\mu})^T/2}$$

where $A$ is a positive-definite symmetric matrix.

Note: With a choice of certain perpendicular main axes, this is just $n$ independent single-variable Gaussians.

## Finding main axes:

An eigenvalue of an $n \times n$ matrix $A$ is a number $\lambda$ for which there exist a column vector $\vec{v} \neq \vec{0}$ such that

$$A\vec{v} = \lambda\vec{v}$$

The vector $\vec{v}$ is called an eigen-vector.

It turns out that if $A$ is a symmetric matrix, it does have $n$ real eigenvalues and the eigenvectors of different eigenvalues are perpendicular – the eigenvectors form the main axes.

The matrix $A$ is positive-definite if and only if all the eigenvalues are positive.

HW: Find $C$ such that

$$C e^{(-4x^2 - xy - 4y^2)/2}$$

is a joint probability density of a 2-variable Gaussian.