MATH 425 11/30/2011

Note Title 11/30/2011

Let $A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$ be an $n \times n$ matrix.

Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exist a complex column non-zero n-dimensional vector $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ such that $A\vec{v} = \lambda \vec{v}$.

The vector $\vec{v}$ is called an eigenvector.

(English: hidden value, hidden vector).

Theorem: When A is a real symmetric matrix then all eigenvalues of A are real and A is positive definite if and only if all the eigenvalues are positive. Additionally, two eigenvectors $\vec{v}, \vec{w}$ belonging to different eigenvalues are orthogonal (= perpendicular, $\vec{v} \cdot \vec{w} = 0$)

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = v_1 w_1 + \dots + v_n w_n .$$

$$\vec{v} \cdot \vec{w} = (\vec{v})^T \vec{w} .$$

Proof: $\overbrace{\text{Complex numbers}}^{\mathbb{C}}$ $x + iy$ $x, y \in \mathbb{R}$, $i^2 = -1$.

The complex conjugate $\overline{x+iy} = x-iy$.

$(x+iy)(\overline{x+iy})$
$= x^2+y^2 \geq 0$
$=0$ if $x+iy=0$

Let $\vec{v}$ be an eigenvector of $\vec{A}$ belonging to an eigenvalue λ ($\lambda, \vec{v}$ possibly complex).

complex conjugate

$$(\overline{\vec{v}}^T)\vec{A}\vec{v} = \overline{\vec{v}}^T\lambda\vec{v} = \lambda\underbrace{(\overline{\vec{v}}^T\vec{v})}_{>0}$$

$$\|$$

$$\left(\overline{\vec{v}}^T\vec{A}\right)\vec{v} = \left(\overline{\vec{v}^T\vec{A}^T}\right)\vec{v} = \overline{(\underbrace{A\vec{v}}_{A\vec{v}=\lambda\vec{v}})}^T\vec{v} =$$

real matrix

$\overline{\vec{A}} = \vec{A}$, symmetric

$\vec{A}^T = \vec{A}$

$$= \bar{\lambda}\underbrace{\overline{\vec{v}}^T\vec{v}}_{>0}$$

$\therefore\ \lambda = \bar{\lambda}$, which means that the eigenvalue is real.

<u>Orthogonality of eigenvectors:</u>

$$A\vec{v} = \lambda\vec{v} \qquad\qquad A\vec{w} = \mu\vec{w}$$

We can assume that $\vec{v}, \vec{w}$ are real vectors.

$$\vec{w}^T(A\vec{v}) = \vec{w}^T\lambda\vec{v} = \lambda(\vec{w}\cdot\vec{v})$$

$$\|$$

$$(\vec{w}^T A)\vec{v} = (\vec{w}^T A^T)\vec{v} = (A\vec{w})^T\vec{v} = \mu\vec{w}^T\vec{v}$$

$$A^T = A \qquad\qquad = \mu(\vec{w}\cdot\vec{v})$$

$$\left.(\underbrace{\lambda-\mu}_{\neq 0})\,\vec{w}\cdot\vec{v} = 0\ \right\}\ \therefore\ \vec{w}\cdot\vec{v} = 0.$$

To show that A is positive definite if and only if all the eigenvalues are positive, I need to know that I can find n linearly independent eigenvectors. (Proved by contradiction, if eigenvectors do not span $\mathbb{R}^n$, look at the orthogonal complement of the space they span.)

If we have n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$, $A\vec{v}_1 = \lambda_1 \vec{v}_1, \ldots, A\vec{v}_n = \lambda_n \vec{v}_n$

Then $\vec{v}_1, \cdots \vec{v}_n$ form a basis of $\mathbb{R}^n$. Let us assume $\vec{v}_1, \cdots \vec{v}_n$ are orthogonal. Then any column vector $\vec{v}$ with n coordinates can be written as $a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n = \vec{v}$.

Then

$$\vec{v}^T A \vec{v} = \underbrace{a_1^2}_{>0} (\lambda_1) \underbrace{\vec{v}_1 \cdot \vec{v}_1}_{>0} + \cdots + \underbrace{a_n^2}_{>0} (\lambda_n) \underbrace{\vec{v}_n \cdot \vec{v}_n}_{>0}.$$

The right hand side is always positive if and only if $\lambda_1, \cdots, \lambda_n > 0$. $\square$

A practical algorithm for finding main axes of a symmetric matrix A (the case of n different eigenvalues):

① Find eigenvalues

$$A\vec{v} = \lambda\vec{v}$$

$$I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots 1 \end{pmatrix} \qquad A\vec{v} = \lambda I \vec{v}$$

$$(A - \lambda I)\vec{v} = 0$$

$$\det(A - \lambda I) = 0$$

(equivalently, $\det(\lambda I - A) = 0$).

an n order polynomial in λ.

Example: $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda - 4 & -1 \\ -1 & \lambda - 4 \end{pmatrix} = (\lambda - 4)^2 - 1 = \lambda^2 - 8\lambda + 15$$

$$\lambda^2 - 8\lambda + 15 = 0 \qquad \lambda_{1,2} = \frac{8 \pm \sqrt{64 - 60}}{2} = \frac{8 \pm 2}{2} = \begin{cases} 5 > 0 \\ 3 > 0 \end{cases}$$

✓ positive definite

$$\begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix}$$

$$\left(5I - \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$x - y = 0$

$x = y$

$$\begin{pmatrix} x \\ x \end{pmatrix}$$

$x^2 + x^2 = 1$

$x^2 = 1/2$

$$\vec{v} = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$$

$$\left(3I - \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)\begin{pmatrix} r \\ s \end{pmatrix} = 0$$

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix} = 0$$

$r + s = 0$

$$\begin{pmatrix} r \\ -r \end{pmatrix}$$

$r^2 + r^2 = 1$

$r^2 = 1/2$

$$\vec{w} = \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix}$$

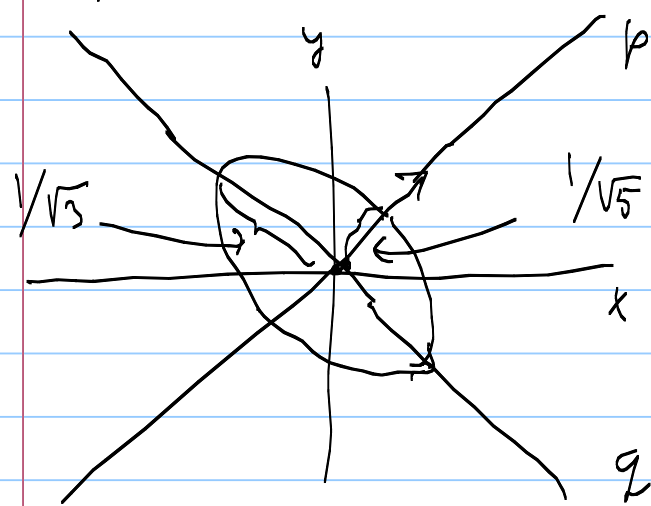
Problem: What is the equation of the ellipse

$$4x^2 + 2xy + 4y^2$$

in main axes, and what are the axes?

Answer: $5p^2 + 3q^2$

The axes are the vectors $\underset{p}{\begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}}$, $\underset{q}{\begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix}}$



Enough about Gaussians.

- sums of independent jointly distributed random variables

- an arbitrary change of variables on jointly distributed random variables

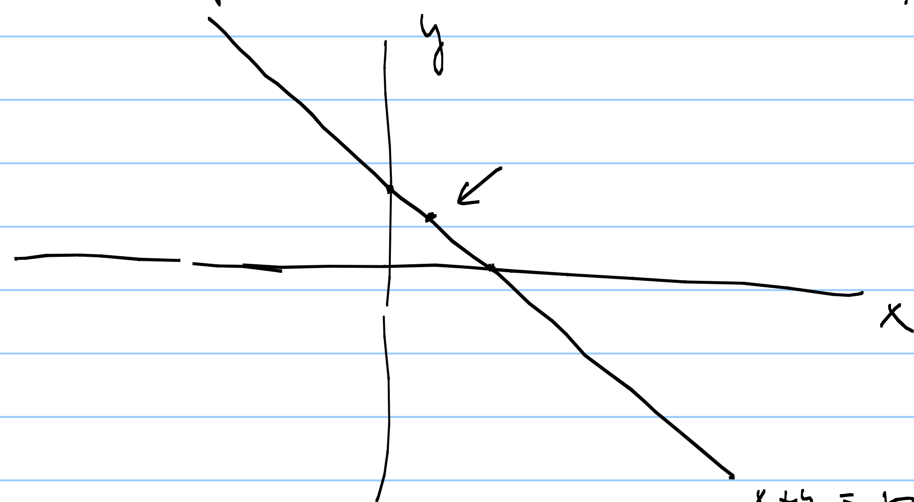
- conditional distributions

Sums

Let X, Y be independent jointly distributed random variables. What is the probability density of $X+Y$?

If x has marginal density $f(x)$

y has marginal density $g(y)$



$x+y=r$

$y=r-x$

$$\boxed{\int_{-\infty}^{\infty} f(x)\, g(r-x)\, dx = \text{density of } X+Y \text{ as a function of } r}$$

HW: ① Find the main axes of the ellipse $41x^2 - 24xy + 34y^2$, and the equation of the ellipse in the main axes.

② Verify that $A = \begin{pmatrix} 66 & -12 \\ -12 & 59 \end{pmatrix}$ is positive definite. Express the Gaussian with joint density

$$\frac{\sqrt{\det A}}{2\pi} e^{-(x,y)A(x,y)^T/2}$$

as two independent 1-variable Gaussians

(variables are the main axes).