MATH 425 12/2/2011

Note Title 12/2/2011

① <u>Sum of independent random variables</u>

6.3 (p. 252) (used when deriving χ^2)

② <u>A general transformation of jointly distributed random variables</u>

6.7 (p. 274) (used when deriving t)

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② For simplicity, consider two jointly distributed random variables X_1, X_2. Suppose we have two other random variables Y_1, Y_2 which are functions of X_1, X_2.

$$Y_1 = g_1(X_1, X_2) \qquad\qquad Y_2 = g_2(X_1, X_2)$$

How do the joint probability densities of (y_1, y_2) and (x_1, x_2) relate? Assume that

$$(y_1, y_2) = (g_1(x_1, x_2), g_2(x_1, x_2)) \qquad (*)$$

is a bijective map $\mathbb{R}^2 \to \mathbb{R}^2$: this means that in $(*)$, every point (y_1, y_2) is the image of precisely one point (x_1, x_2). Let the density function of (X_1, X_2) be $f_{X_1, X_2}(x_1, x_2)$.

From the point of view of integration, the density is actually

$$f_{X_1,X_2}(x_1,x_2)\,dx_1\,dx_2 .$$

Let the inverse formula of $(*)$ be

$$(x_1,x_2) = (h_1(y_1,y_2),\ h_2(y_1,y_2)).$$

The integral density of (Y_1,Y_2) is

$$f_{X_1,X_2}(h_1(y_1,y_2),\ h_2(y_1,y_2))\,dx_1\,dx_2 \qquad (**)$$

So we need to relate $dy_1\,dy_2$ to $dx_1\,dx_2$ to get the density function: (Calc. 3 - Jacobian formula)

$$dy_1\,dy_2 = \left|\det\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{pmatrix}\right| dx_1\,dx_2 \qquad \boxed{\begin{array}{l}\text{by } (*) \\ y_1 = g_1(x_1,x_2) \\ y_2 = g_2(x_1,x_2)\end{array}}$$

From $(**)$, the integral density of (Y_1,Y_2) is

$$f_{Y_1,Y_2}(y_1,y_2)\,dy_1 dy_2 =$$

$$= f_{X_1,X_2}(h_1(y_1,y_2), h_2(y_1,y_2)) \left(\frac{\partial g_1}{\partial x_1}\cdot\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1}\right)^{-1} dy_1 dy_2.$$

We conclude that the joint probability density <u>function</u> of (Y_1, Y_2) is

$$\boxed{\begin{aligned} & f_{X_1,X_2}(h_1(y_1,y_2), h_2(y_1,y_2)) \underbrace{\left(\frac{\partial g_1}{\partial x_1}\frac{\partial g_2}{\partial x_2} - \frac{\partial g_1}{\partial x_2}\frac{\partial g_2}{\partial x_1}\right)^{-1}}_{J^{-1}} \\ & = f_{Y_1,Y_2}(y_1,y_2) \end{aligned}}$$

<u>Example</u>: A sum of random variables X_1, X_2:

$$Y_1 = X_1 + X_2 \qquad \frac{\partial y_1}{\partial x_1} = 1 \qquad \frac{\partial y_2}{\partial x_2} = 1$$

$$Y_2 = X_2 \qquad \frac{\partial y_1}{\partial x_2} = 1 \qquad \frac{\partial y_2}{\partial x_1} = 0$$

$$J^{-1} = (1\cdot 1 - 1\cdot 0)^{-1} = 1$$

How to recover X_1, X_2 from Y_1, Y_2?

$$X_1 = Y_1 - Y_2$$

$$X_2 = Y_2$$

note $J^{-1} = 1$.

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}(y_1-y_2, y_2)$$

The probability density of $X_1 + X_2 = Y_1$ is the Y_1-marginal density for Y_1, Y_2:

$$f_{X_1+X_2}(y_1) = \int_{y_2=-\infty}^{\infty} f_{Y_1,Y_2}(y_1,y_2)\,dy_2 = \int_{y_2=-\infty}^{\infty} f_{X_1,X_2}(y_1-y_2,y_2)\,dy_2$$

Denoting $y_1 = x$, $y_2 = t$:

$$\boxed{f_{X_1+X_2}(x) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x-t,t)\,dt}.$$

If X_1, X_2 are <u>independent</u>, then $f_{X_1,X_2}(x-t,t) = f_{X_1}(x-t)\,f_{X_2}(t)$

so $$\boxed{f_{X_1+X_2}(x) = \int_{-\infty}^{\infty} f_{X_1}(x-t)\,f_{X_2}(t)\,dt}$$

See Section 6.3 for many examples.

In particular,

- The sum of independent Gaussians X_1, X_2 is a Gaussian

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$ } Expectation is always additive

$$\mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2)$$ } variance is additive for independent variables.

<u>Recall</u>: variance behaves like the square length of a vector. $\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2u\cdot v$

For <u>general</u> jointly distributed random variables

X_1, X_2,

$$\mathrm{var}(X_1 + X_2) = \mathrm{var}(X_1) + \mathrm{var}(X_2) - 2\,\mathrm{cov}(X_1, X_2)$$

the <u>covariance</u>

$$\boxed{\mathrm{cov}(X_1, X_2) = E(X_1 \cdot X_2) - E(X_1)\,E(X_2).}$$

- Also, for binomial distributions $X_{m,p}$, $X_{n,p}$

note: <u>discrete</u>

trials

probability of success

if independent:

$$X_{m,p} + X_{n,p} = X_{m+n,p}$$

- Also for the χ^2-distribution: if

$$X_1 = \underbrace{\chi^2 \text{ with } k \text{ degrees of freedom}}_{\text{a sum of } k \text{ independent standard Gaussians}}$$

$$X_2 = \chi^2 \text{ with } l \text{ degrees of freedom}$$

$$X_1 + X_2 = \chi^2 \text{ with } k+l \text{ degrees of freedom}$$

if X_1, X_2 independent.

<u>Note</u>: χ^2 is a special case of Γ with $\lambda = \frac{1}{2}$. The same is true for a general Γ when X_1, X_2 have the same λ.

Conditional distributions

For events E, F $\qquad P(E|F) = \frac{P(E\cap F)}{P(F)}$ if $P(F) > 0$.

For random variables X, Y, the conditional distribution function is

$$\boxed{F_{X|Y}(x|y) = P\{X \le x \mid Y = y\}.}$$

This is defined for any X, Y, but only almost everywhere with respect to Y, i.e. outside of a measurable set B of real numbers such that

$$P\{Y \in B\} = 0.$$

For discrete variables:

$$F_{X|Y}(x|y) = \sum_{a \leq x} p_{X|Y}(a|y)$$

$$p_{X|Y}(a|y) = \frac{P(\{X=a\} \cap \{Y=y\})}{P\{Y=y\}}$$

joint probability mass function

denominator: probability mass function.

For continuous jointly distributed random variables

X, Y: $\quad F_{X|Y}(x|y) = \int_{t=-\infty}^{x} f_{X|Y}(t|y)\, dt$

$$\boxed{f_{X|Y}(t|y) = \frac{f_{X,Y}(t,y)}{f_Y(y)}}$$

HW: 1 Find a formula for the probability density function of $\frac{X}{Y}$ $(Y>0)$ where X, Y are jointly distributed random variables in terms of the joint probability density of X, Y.

2 6.40 p.289