Math 425 12/5/2011

Note Title 12/5/2011

Example 5a p. 266

X, Y are jointly continuous random variables

joint density:

$$f(x,y) = \begin{cases} \frac{12}{5} x(2-x-y) & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{else} \end{cases}$$

Compute the conditional density of X at Y = y, $0 < y < 1$.

Solution:

Computing the marginal density:

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{12}{5} x(2-x-y)\,dx$$

$$= \frac{12}{5}x^2 - \frac{4}{5}x^3 - \frac{6}{5}x^2 y \Big|_0^1 =$$

$$= \frac{8}{5} - \frac{6}{5}y$$

<u>Answer</u>: $$\frac{f(x,y)}{f(y)} = \frac{\frac{12}{\cancel{5}}x(2-x-y)}{\frac{8}{\cancel{5}} - \frac{6}{\cancel{5}}y} = \frac{6x(2-x-y)}{4-3y}.$$

Example 56 p. 267 Suppose X, Y have joint density

$$f(x,y) = \begin{cases} \dfrac{e^{-x/y} e^{-y}}{y} & 0 < x < \infty, \ 0 < y < \infty \\ 0 & \text{else.} \end{cases}$$

Find $P\{X > 1 \mid Y = y\}$.

Solution: Marginal density

$$f(y) = \int_0^\infty \frac{e^{-x/y} e^{-y}}{y} \, dx = - e^{-x/y} e^{-y} \Big|_0^\infty =$$

$$= e^{-y}$$

Conditional density:

$$\frac{f(x,y)}{f(y)} = \frac{e^{-x/y} e^{-y}/y}{e^{-y}} = \frac{e^{-x/y}}{y}$$

Answer:

$$\int_1^\infty \frac{e^{-x/y}}{y}\,dx = -\frac{y\,e^{-x/y}}{y}\Bigg|_{x=1}^{\infty} = \underline{\underline{e^{-1/y}}}.$$

The limit theorems

Roughly speaking, the strong law of large numbers says that when we perform independent trials which are random variables with the same distribution and that the expected value exists, then the average converges to the expected value almost certainly

To phrase the law mathematically:

We have a sample space S with a probability P.

In addition, we have a sequence of random variables $X_1, X_2, X_3, \ldots, X_n, \ldots$. $(X_n : S \to \mathbb{R})$.

We assume that X_n have the same cumulative distribution:

$$P\{X_n \leq a\} = P\{X_1 \leq a\} \text{ for all } a \in \mathbb{R}.$$

We assume that $E(X_1)$ $(= E(X_n))$ exists.

$$\int_S X_n \, dP \quad \text{exists} \qquad \left(\int_S |X_n| \, dP < \infty \right)$$

(In the discrete case: $\sum_{a \in \mathbb{R}} |a| \cdot P\{X_n = a\} < \infty$.

In the continuous case: $\int_{-\infty}^{\infty} x f_X(x) \, dx$.)

We assume also that X_n are independent: If $B_1, \dots B_m$ are measurable subsets of S then

$$P(\{X_1 \in B_1\} \cap \{X_2 \in B_2\} \cap \cdots \cap \{X_n \in B_n\}) =$$

$$= P\{X_1 \in B_1\} P\{X_2 \in B_2\} \cdot \ldots \cdot P\{X_n \in B_n\}$$

for every n. Then

$$P\{\underbrace{\lim_{n\to\infty} \frac{1}{n}(X_1 + \cdots + X_n)}_{\text{limit exists and is equal to } E(X_1)} = E(X_1)\} = 1.$$

Central limit theorem

If we have a sequence of identically distributed random variables X_n which are independent such that the expected value μ exists and the variance σ^2 is finite. Then the <u>distribution</u> of

$$\sqrt{n}\left(\left(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right)\right)$$

expected value 0

converges to the centered Gaussian with variance σ^2.

HW: (1) 6.34 p. 289 (3) 6.55

(2) 6.39