MATH 425 12/9/2011

Note Title 12/9/2011

Review

Continuous random variables

$$P\{X \leq a\} = \int_{-\infty}^{a} f_X(x)\,dx$$

$$f_X(x) \geq 0$$

$f_X(x)$ = probability density

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$\mathrm{var}(X) = E(X^2) - E(X)^2$$

Examples:

Uniform distribution

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \end{cases} \qquad a < b$$

(1-variable)

Gaussian (normal) distribution: Characterised by expected value μ and variance σ^2

$$f_X(x) = \frac{1}{(\sqrt{2\pi})\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Standard Gaussian: $\mu = 0$, $\sigma^2 = 1$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The exponential random variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

The hazard rate function of a random variable X

$$\lambda(x) = \frac{f_X(x)}{1 - \int_{-\infty}^{x} f_X(t)\, dt}$$

(The exponential variable has constant hazard rate λ for $x > 0$.)

The χ^2 random variable with k degrees of freedom

$$f_X(x) = \begin{cases} \dfrac{1}{2^{k/2}\,\Gamma(k/2)}\, x^{k/2-1}\, e^{-x/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\Gamma(\alpha) = \int_0^\infty e^{-x}\, x^{\alpha-1}\, dx.$$

$\alpha > 0$ integer $\Rightarrow$ $\Gamma(\alpha) = (\alpha - 1)!$

Generally: $\Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1).$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

The Student t with k degrees of freedom

$$\boxed{\frac{\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{k\pi}\;\Gamma\left(\frac{k}{2}\right)}\left(1+\frac{t^2}{k}\right)^{-\frac{k+1}{2}} = f_X(t)}$$

$k=1$ Cauchy

To get to the stories of the distributions, we need to talk about joint (multivariable) distributions: $X_1, \ldots, X_n : S \longrightarrow \mathbb{R}$

Discrete — only countably many values:

joint probability mass function

$$p_{X_1,\dots X_n}(x_1,\dots x_n) = P\{X_1=x_1\}\cap\dots\cap\{X_n=x_n\}.$$

Example: The multinomial random variables

with probabilities $p_1,\dots p_n$, $p_1+\dots+p_n=1$

N trials

$$p_{X_1,\dots X_n}(N_1,\dots,N_n) = \binom{N}{N_1,\dots N_n} p_1^{N_1}\cdot\ \dots\ \cdot p_n^{N_n}.$$

jointly continuous: $\exists$ joint probability density

$$f_{X_1,\dots,X_n}(x_1,\dots x_n)$$

$$P\{(X_1, \dots X_n) \in B\} = \int \cdots \int_B f_{X_1, \dots X_n}(x_1, \dots x_n)\, dx_1 \cdots dx_n$$

measurable subset of $\mathbb{R}^n$

Omit one variable:

$$f_{X_1, \dots X_{n-1}}(x_1, \dots x_{n-1}) = \int_{-\infty}^{\infty} f_{X_1, \dots X_n}(x_1, \dots x_n)\, dx_n.$$

Substitution theorem: If $Y_1, \dots, Y_n$ are a bijective function of $X_1, \dots X_n$

$$dy_1 \cdots dy_n = |J|\, dx_1 \cdots dx_n$$

$$J = \det \begin{pmatrix} \partial y_1/\partial x_1 & \cdots & \partial y_1/\partial x_n \\ \vdots & & \\ \partial y_n/\partial x_1 & \cdots & \partial y_n/\partial x_n \end{pmatrix}$$

$$f_{Y_1, \dots Y_n}(y_1, \dots y_n) = f_{X_1, \dots X_n}(x_1(y_1 \dots y_n), \dots x_n(y_1, \dots y_n)) \cdot |J|^{-1}$$

<u>Independent</u> variables $X_1, \dots, X_n$

$$P(\{X_1 \in B_1\} \cap \dots \cap \{X_n \in B_n\}) = P\{X_1 \in B_1\} \cdot \dots \cdot P\{X_n \in B_n\}.$$

Stories: For any two jointly distributed random variables X, Y

$$E(X+Y) = E(X) + E(Y)$$ (if the right hand side exist)

If X, Y are independent then

$$\text{var}(X+Y) = \text{var}(X) + \text{var}(Y).$$

Gaussian story:

The sum of independent Gaussians is a Gaussian (E and var are additive).

Central limit theorem: X_i are independent, have the same distribution, variance σ^2 and expected value μ then the distribution of

$$\frac{\sqrt{n}}{\sigma}\left(\frac{X_1+\dots+X_n}{n}-\mu\right)$$

converges to the distribution of the standard Gaussian.

Multivariable Gaussian A positive-definite symmetric matrix

$$\frac{\sqrt{\det A}}{(2\pi)^{n/2}} \, e^{-\vec{x}^T A \vec{x}/2} = f_{\vec{X}}(\vec{x})$$

$\leftarrow$ covariance matrix

$$\vec{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}.$$

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y)$$

($\mathrm{cov}(X,Y) = 0$ if X, Y are independent).

<u>Story of χ^2</u>

$X_1, \dots X_k$ = independent standard Gaussians.

$$\chi^2 = X_1^2 + \dots + X_k^2$$

Pearson test: testing an observed sampling against the multinomial distribution:

N trials n variables $X_1, \ldots X_n$ expected for X_i is $p_i N$, Observed $= O_i$

$$\sum_{i=1}^{n} \frac{(O_i - p_i N)^2}{p_i N} = \chi^2_{\underbrace{n-1}}$$

$n-1$ degrees of freedom.

Bayesian analysis: can reject the hypothesis with certainty $C\%$ (95%) if $\chi^2 >$ C percentile

Student t story : $T = \dfrac{Z\sqrt{k}}{\chi^2_k}$

Z = standard normal

χ^2_k = χ^2 with k degrees of freedom

Z, χ^2_k are independent.

$X_1, \dots, X_n$ independent random variables
same distribution (assume normal)

testing the expected value against a number μ_0

$$T = \frac{\frac{1}{n}(X_1 + \cdots + X_n) - \mu_0}{\sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n(n-1)}}}$$

$$\overline{X} = \frac{1}{n}(X_1 + \cdots + X_n)$$

degrees of freedom: $k = n-1$.

(One-tailed test $>$ mean

alt. $<$ mean) $\leftarrow$ 95%

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two-tailed test $\neq$ mean. $\leftarrow$ 90%