Let $C, D$ be categories, $F, G : C \to D$ be functors.

A natural transformation $\eta : F \to G$ is a collection of morphisms in $D$

$\eta_x : F(x) \to G(x), \quad x \in \text{Obj } C$

such that for a morphism $f : x \to y \in \text{Mor } C$, we have a commutative diagram

$\begin{array}{ccc}
F(x) & \xrightarrow{F(f)} & G(x) \\
\downarrow{\eta_x} & & \downarrow{\eta_y} \\
F(y) & \xrightarrow{F(g)} & G(y)
\end{array}$
In a category $\mathcal{C}$, an isomorphism is a morphism $f : X \to Y$ such that there exist a morphism $g : Y \to X$ with $gf = \text{Id}_Y$, $fg = \text{Id}_X$. (We usually write $g = f^{-1}$.)

Remark: If $\eta : F \to C$ and each $\eta_x$, $x \in \text{Obj } \mathcal{C}$ is an isomorphism, then $\eta^{-1}_x \left( \left( \eta_x^{-1} \right)_x \right)$ is natural.

Example: In the category of $R$-modules, if $R$ commutes with direct sums, we have natural isomorphisms

\[
\begin{align*}
A \oplus_R R &\cong A & R \oplus_R A &\cong A \\
A \otimes_R B &\cong B \otimes_R A & A \otimes_R (B \otimes_R C) &\cong (A \otimes_R B) \otimes_R C.
\end{align*}
\]
Note: There are certain diagrams which should commute if we want to use these monomorphisms to assert that the is commutative and associative. 

we can sometimes process a word to another word in two different ways.)

something you can write on symbols using your identities using your operation

\[ a(b(cd)) = a((bc)d) = (a(bc))d = ((ab)c)d \]

\[ (ab)(cd) \] coherence diagrams
\[
\begin{align*}
A \otimes_R (B \otimes_R (C \otimes_R D)) & \cong A \otimes_R (B \otimes_R C) \otimes_R D \\
& \cong (A \otimes_R B) \otimes_R C \\
& \cong (A \otimes_R B) \otimes_R (C \otimes_R D) \\
& \cong (A \otimes_R (B \otimes_R C)) \otimes_R D
\end{align*}
\]

**Examples and properties of \( \otimes_R \)**

\[
\mathbb{Z}/m\mathbb{Z} \otimes_R \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}
\]

\[
m | \otimes_1 = 0 \quad 1 \otimes_1 = 1
\]

\[
m | \otimes 1 = 1 \otimes m = 0 \quad a \otimes b \mapsto ab
\]

\[
\gcd(m,n) | \otimes_1 = 0
\]
\( \otimes_k \) is distributive under \( \bigoplus \)

\[
\left( \bigoplus_{i \in I} M_i \right) \otimes_k N \cong \bigoplus_{i' \in I} (M_{i'} \otimes_k N)
\]

\[\text{inclusion } \otimes_k N\]

A bilinear map

\[
\left( \bigoplus_{i \in I} M_i \right) \times N \rightarrow \bigoplus_{i' \in I} (M_{i'} \otimes_k N)
\]

\[
\sum_{i \in I} (m_i \otimes n) \rightarrow \sum_{i' \in I} (m_{i'} \otimes n)
\]

0 unless \( i' \in F \) for some

\( (i' \in F) \)
\[ R(S \times T) = R_S \otimes_R R_T. \]

Let \( S, T \) set mutually

(If \( R \) is a field, the tensor product of two vector spaces of dimensions \( m, n \) respectively has dimension \( mn \).)

\[ \text{Let } R_i \xrightarrow{\varphi} R_i \text{ be a homomorphism of commutative rings. Then for an } R_i \text{-module } M, R_i \otimes_{R_i} M \text{ has canonically a structure of an } R_i \text{-module.} \]
and satisfies the following universal property:

Note that:

\[ 
\begin{array}{ccc}
\Pi &=& R_1 \otimes_{R_1} P \\
\uparrow & \sim & \downarrow \\
& & R_L \otimes_{R_1} P
\end{array}
\]

This is called base change.
(Extension of scalars)

Exactness properties: If

\[ 0 \to M_1 \xrightarrow{\alpha} M \xrightarrow{\beta} M_2 \to 0 \]

is a short exact sequence of \( R \)-modules and \( P \) is an \( R \)-module then the following sequence of \( R \)-modules is exact (Right Exactness)

\[ 0 \to \left( M_1 \otimes_R P \right) \xrightarrow{\alpha \otimes 1} \left( M \otimes_R P \right) \xrightarrow{1 \otimes \beta} \left( M_2 \otimes_R P \right) \to 0 \]

Why can't we have a 0 on the left?

Example: \( R = \mathbb{Z} \)

\[ 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0 \]
(What goes to the left?): not injective!

Introduction to algebra, topology

Homological algebra

\[ \mathbb{Z} \twoheadrightarrow \mathbb{Z} \twoheadrightarrow \mathbb{Z} \twoheadrightarrow 0 \]

Homological Algebra: \( \beta \otimes \mathbb{F}_p \to \text{out} \)

\[ \mathbb{F}_p \otimes \mathbb{F}_p \to \mathbb{F}_p \otimes \mathbb{F}_p \to 0 \]

(\( \beta \otimes 0 \)) generate as an \( R \)-module

\[ \text{Coh } \mathbb{F}_p \quad \mathbb{F}_p \]

\[ \mathbb{F}_p \quad \mathbb{F}_p \]
Exactness at $\Pi \otimes_\mathbb{K} p$ needed to prove

$$\Pi_2 \otimes p \hookrightarrow \Pi \otimes p / \operatorname{Im}(\alpha \otimes_\mathbb{K} \text{Id}_p)$$

homomorphism $\tilde{\alpha} \in \ker \beta \otimes_\mathbb{K} \text{Id}_p$

Theorem

$$(\beta \otimes_\mathbb{K} \text{Id}_p)(\alpha \otimes_\mathbb{K} \text{Id}_p)$$

$$= \beta \alpha \otimes_\mathbb{K} \text{Id}_p = 0.$$
HW: Prove that if $A$ is a torsion abelian group ($\forall a \in A \exists n \in \mathbb{N} - \{1, 2, \ldots\} \; na = 0$), then $\mathbb{Z} \otimes A = 0$. See Lecture for other problems!

[3 Problems Total]


1 - 2 + Extra Hour

If Needed