Examples: \( R = \Lambda_Q [x] = Q[x]/(x^2) \).

\( Q \) is an \( R \)-module where \( x \) acts by 0.

\[
0 \to Q \to \Lambda_Q [x] \to Q \to 0
\]

long exact sequence

\[
\cdots \to \Lambda_Q [x] \overset{x}{\rightarrow} \Lambda_Q [x] \overset{x}{\rightarrow} \Lambda_Q [x] \overset{x}{\rightarrow} \Lambda_Q [x] \overset{x}{\rightarrow} Q \to 0
\]
A free resolution of the $\Lambda Q[x]$-module $Q$:

$$
\cdots \rightarrow \Lambda Q[x] \xrightarrow{x} \Lambda Q[x] \xrightarrow{\delta_3} \Lambda Q[x] \xrightarrow{\delta_2} \Lambda Q[x] \xrightarrow{\delta_1} \Lambda Q[x] \rightarrow C
$$

$$
\text{Hom}_{\Lambda Q[x]}(Q, Q) = 0 \quad \forall \ i > 0
$$

$$
\text{Ext}_{\Lambda Q[x]}^i(Q, Q) = 0 \quad \forall \ i \in \mathbb{N}_0
$$

Remarks: ① If $R = F$ is a field, every
module is free (every vector space has a basis)

\[ \text{Tor}_n^R(, ) = 0, \quad \text{Ext}_n^R = 0 \quad n > 0. \]

2. If \( R \) is a PID then a submodule of a free module is free, \( V_R \)-module \( M \)

\[ 0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0 \]

\[ \text{free } R\text{-module} \quad \text{free } R\text{-module} \]

\[ (R \text{ factor}_R M^R) \]

\[ K \rightarrow F \quad \text{is a free resolution of } M \]

\[ \text{Tor}_n^R = \text{Ext}_n^R = 0 \quad \text{for } n > 1. \]
D: \( \text{(3) \hspace{1cm} \mathfrak{R} = \Lambda_{\alpha} \mathfrak{L}[x] } \)

\[ \Lambda_{\alpha}[x] \xrightarrow{x} \Lambda_{\alpha}[x] \xrightarrow{\cdot x} \ldots \Lambda_{\alpha}[x] \xrightarrow{\cdot x} \ldots \]

This is an exact sequence, so it satisfies the definition of a resolution of 0 except \( C_i = 0 \) for \( i < 0 \).

Hence, \( \Lambda_{\alpha}[x] (0, \alpha) \) has cohomology 0 in every dimension.

\[ \cdots \Omega^0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \cdots \Omega^0 \xrightarrow{d} \cdots \]

So the assumption \( C_i = 0 \) for \( i < 0 \) cannot be dropped from the definition of a resolution!
HW: Consider $Q$ as a module over $Q[x, y]$.

$x$ and $y$ act by 0 on $Q$. Compute

$\text{Tor}_1^Q(x, y) \times (Q, Q)$, $\text{Ext}_1^Q(x, y) \times (Q, Q)$.

Long exact sequences

on $\text{Tor}$, $\text{Ext}$

from short exact sequences of $R$-modules.

$0 \to M' \xrightarrow{2} M \xrightarrow{k} \to M'' \to 0$

be a short exact sequence of $R$-modules.
Long exact sequences: Let $p$ be an $R$-module.

\[ 0 \rightarrow \text{Tor}_1^R(p, N) \rightarrow \text{Tor}_i^R(p, N) \rightarrow \text{Tor}_i^R(p, N') \rightarrow \]

\[ \cdots \]

(1)

(\text{In symmetric, so also in the other variable).)

Proof:

Let $C$ be a free $R$-resolution of $p$.

\[ 0 \rightarrow C \otimes_R N' \rightarrow C \otimes_R N \rightarrow C \otimes_R N'' \rightarrow 0 \]

$C$ is a free resolution! $C_i = RS_i$, a free $R$-module.
\[ C_i = \bigoplus_{s \in s_i} R \]

\[ \otimes_R \text{ preserves sums, } \]

\[ R \text{ is the unit under } \otimes_R \]

exactness of \( \otimes \) follows. \( \Rightarrow \) Pass to long exact sequence in homology. We get \( C_i \).

It works similarly for \( \text{Ext}^i \)

\[ 0 \to \text{Hom}_R (C_i, M') \to \text{Hom}_R (C_i, M) \to \text{Hom}_R (C_i, M') \to 0 \]

homomorphisms are specified by values on generators.
$C_i = R^S_i$

$0 \to \prod_{s \in S_i} M' \to \prod_{s \in S_i} M \to \prod_{s \in S_i} M'' \to 0$

$\prod_{s \in S_i}$

$\prod$ preserves exactness. $\Rightarrow (2)$.

$(\exists)$ $\Rightarrow \text{Ext}^i_R(M''', p) \xrightarrow{k^*} \text{Ext}^i_R(M', p) \xrightarrow{z^*} \text{Ext}^i_R(M_i, p) \xrightarrow{f} \text{Ext}^i_R(n''', p) \xrightarrow{}$

$z^* = \text{Hom}_R (2, \text{Id}_p)$

Invariant.

To prove $(3)$, we need projective resolutions.
An $R$-module $C$ is called projective if it is a direct summand of a free module.

If $D$ is an $R$-module and a set $S$ and an isomorphism $C \otimes D \cong RS$, then $\text{Tor}_i^R(C, D)$ is projective.

Lemma: Projective resolutions may also be used to calculate $\text{Tor}_i^R(C, D)$.

Proof: Consider a projective resolution:

$$C : \ldots \to C_2 \to C_1 \to C_0 \to 0 \ldots$$
I can add (with respect to \( \oplus \)) complexes of the form

\[
0 \to P \to P \to 0 \tag{chain homotopy equivalent to 0}
\]

to make a free resolution. (chain contractible)

\[
\cdots \to C_2 \to C_1 \to C_0 \to 0
\]

\[
\oplus\quad \oplus \quad \oplus \quad \oplus \quad \oplus
\]

\[
C_0 \to C_0 \tag{free}
\]

\[
P_1 \to P_1
\]

I produced a free resolution chain - homotopy equivalent to \( C \). But we already observed that...
chain homotopy (hence chain homotopy equivalence) is preserved by $\otimes R M$

$\text{Hom}_R (\mathbb{Z}, M)$.

So I will get the same $\text{Tor}_i^R \text{Ext}_R^j$ from the projective resolution.

To get a long exact sequence for $\text{Ext}_R^j$ in the first variable, I need to produce a short exact sequence of resolutions:

$$(+) \quad 0 \rightarrow \mathcal{N}^1 \rightarrow \mathcal{N} \rightarrow \mathcal{N}^2 \rightarrow 0$$
$0 \to C' \to C \to C'' \to 0$ by resolutions which, in $\mathcal{H}_0$, gives $C^1$:

- Take any resolution
- Add a free resolution
- $0 \to 0 \to 0$
- $0 \to 0 \to 0$
- $0 \to 0 \to 0$
- $0 \to 0 \to 0$
- $0 \to 0 \to 0$

Kernels $\to K_0 \to K_0 \to K_0$

$0 \to M' \to M \to M'' \to 0$
\[ \begin{align*}
\mathbb{R}^n & \xrightarrow{\theta} \mathbb{R}^n \\
\text{Id} & \downarrow \\
\mathbb{R}^n & \xrightarrow{\varphi} \mathbb{R}^n \\
\mathbb{R}^n & \xrightarrow{\text{Id}} \mathbb{R}^n
\end{align*} \]

\[ \mathbb{R}^n \xrightarrow{\theta} \mathbb{R}^n \]

\[ \Phi \]

\[ 0 \quad \text{def} \quad k_i \xrightarrow{c_i} c_i'' \xrightarrow{0} \]

\[ \Phi \]

\[ \text{id} \]

\[ c_i'' = R s_i \]

\[ \therefore c_i = k_i \circ c_i'' \]

define the splitting on the

\[ k_i \circ \text{proietto}, \quad \text{rel} \quad s_i'' \]
\[(\ast) \quad 0 \to K \to C \to C' \to 0\]

\[\text{Im } K_{0} \colon 0 \to \mathfrak{n}' \to \mathfrak{n} \to \mathfrak{n}' \to 0\]

\[\text{Hom } (C_{k}) \rho \text{ is still short exact.}\]
\[\uparrow\]
\[0 \to K_{i} \to C_{i} \to C_{i}' \to 0\]

commutes with
\[? \Theta ?\]
\[C_{i} = K_{i} \circ C_{i}'; \overline{\rho} \text{ lift}\]

Apply \(H^{\ast}(?)\) get long exact sequence in cohomology.
Symmetry of Tor


\textbf{Theorem}: There is a natural isomorphism

\[ \text{Tor}_n^R(\mathcal{M}, N) \cong \text{Tor}_n^R(N, \mathcal{M}). \]

Why can't we take free resolutions both of \( \mathcal{M}, N \)? How they play symmetrical roles?

But what if \( \mathcal{C} \otimes_R D \) of chain complexes of \( R \)-modules?

\textbf{Caution}: \( \to C_n \otimes_R \mathcal{M} \xrightarrow{d_{n+1}} C_{n-1} \otimes_R \mathcal{M} \to \) is the wrong construction!
that construction does not preserve chain homotopy equivalence (for example).

\[
\begin{array}{c}
\begin{pmatrix} R \\ \text{Id} \\ R \end{pmatrix} & R & R \otimes R & R \\
\begin{pmatrix} 1 \\ 0 \end{pmatrix} & 0 & 0 & 0
\end{array}
\]

Only go down by 1 in dimension: a double complex:

\[
\begin{array}{c}
\begin{pmatrix}
\text{Id} & d \\
\text{Id} & d \\
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix}
C_n \otimes R & C_{n-1} \otimes R \\
C_n \otimes R & C_{n-1} \otimes R
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix}
C_n \otimes R & C_{n-1} \otimes R \\
C_n \otimes R & C_{n-1} \otimes R
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix}
\text{Id} & d \\
\text{Id} & d
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix}
C_n \otimes R & C_{n-1} \otimes R \\
C_n \otimes R & C_{n-1} \otimes R
\end{pmatrix}
\end{array}
\]

\[
\begin{array}{c}
\begin{pmatrix}
\text{Id} & d \\
\text{Id} & d
\end{pmatrix}
\end{array}
\]
the square commutes.

This is an example of a **double chain complex**:

\[
C = \left( C_{m,n} \right) \quad \text{R-modules}
\]

**definition**

\[
\partial : C_{m,n} \to C_{m-1,n} \quad \partial \partial = 0
\]

**double chain complex**

\[
\delta : C_{m,n} \to C_{m,n-1} \quad \delta \delta = 0
\]

**of**

\[
\exists \ \mathfrak{D} = \mathfrak{D}_0
\]

**R-modules**

**double chain complex**

\[
\text{totalization}
\]
If \( C \) is a double chain complex then
the totalization \( |C| \) is a chain complex
defined as follows:

\[
|C|_n = \bigoplus_{k \in \mathbb{Z}} C_{k, n-k}
\]
Define for $C, D$ chain complexes of $R$-modules,

$$C \otimes_R D = \left\{ \sum c_i \otimes d_i \in C \otimes_R D \mid \sum c_i = 0 \right\}$$

as \text{totalization of a double chain complex}.

- a functor
- preserves short exact sequences in either variable which are split in each dimension.
An isomorphism

\[ C \otimes_R D \longrightarrow D \otimes_R C \]

Claim: Let \( C \) resp. \( D \) be a free resolution of an \( R \)-module \( M \) resp. \( N \). Then

\[ H_n(C \otimes_R D) \cong \text{Tor}_n^R(M, N). \]
\[ \partial : C^m \rightarrow C^{m-1} \quad \delta : C^m \rightarrow C^{m-1} \]

such that

(1) \( C^m \rightarrow 0 \) for \( m < 0 \)

(2) \( \cdots \rightarrow C^m \rightarrow C^{m-1} \rightarrow C^{m-2} \rightarrow \cdots \)

is exact (has 0 homology) for each \( m \).

Then \( H_m(\mathbb{C}) = 0 \) \( \forall m \in \mathbb{Z} \).
Why do I care? The augmented resolution of $M$ is

$$\cdots \to C_3 \to C_2 \to C_1 \to M \to 0 \to \tilde{C}$$

$$\text{Id} \hspace{1cm} \text{Id} \hspace{1cm} \text{Id} \hspace{1cm} \text{Id} \hspace{1cm} \text{Id} \hspace{1cm} \text{Id} \hspace{1cm} \text{Id} \hspace{1cm} \text{Id}$$

$$\begin{array}{cccccccc}
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\text{Id} & \text{Id} & \text{Id} & \text{Id} & \text{Id} & \text{Id} & \text{Id} & \text{Id} \\
\end{array}$$

$$\text{dimannin}$$

$$\cdots \to C_3 \to C_2 \to C_1 \to C_0 \to 0 \to \tilde{C}$$

Short exact sequence:

$$0 \to \mathbb{P}[-1] \to \tilde{C} \to C \to 0$$

Apply $? \otimes_R D$ where $D$ is a free resolution of $M$.

exact because $D$ is free

(+) $$0 \to M \otimes_R D[-1] \to \tilde{C} \otimes_R D \to C \otimes_R D \to 0$$
But $H^m(\widetilde{C} \otimes R D)$ has 0 homology by the technical lemma. Take the long exact sequence in homology applied to (E):

$$H^m(\widetilde{C} \otimes R D) \rightarrow H^m(C \otimes R D) \rightarrow H^m(\Pi \otimes R D) \rightarrow 0$$

$\mapsto$

$$H^m(D \otimes R C)$$

$\mapsto$

$\mapsto$

$$H^m(\Pi \otimes R D)$$

$\mapsto$

$\mapsto$

$$T^R_{\Pi}(\Pi, N)$$

$\mapsto$

$$T^R_{\Pi}(\Pi, N)$$
Proof of the technical lemma:

The length of a cycle \( c = (c_{mn})_{m,n \in \mathbb{Z}} \) is \( \sum c_{mn} \).

But only finitely many non-zero is \( \max \{ n \mid c_{mn} \neq 0 \} \).

For contradiction, take a cycle \( c \) which is not a boundary of minimal height \( m \).

Let \( G \) be in dimension \( p \).
\[ \partial c_{p-n, n} = 0 \quad \text{But rows are exact.} \]

So

\[ c_{p-n, n} = \partial x \left( p-m+1, m \right) \]

The two dimensions

So replace

\[ c \quad \text{by} \quad c - d \alpha (p-m+1, m) \]

only one non-zero is

lower bound = contradiction.

Let's go back to \( V \): let \( C \) be chain.
complexes of free abelian groups. I can figure out $H_* (C \otimes D)$ from $H_* C$ and $H_* D$.

Reason: Structure Theorem:

$$C \cong \bigoplus_{m \in \mathbb{Z}} X_m [m]$$ where $X_m$ is a free $\mathbb{Z}$-resolution of $H_m C$.

$$D \cong \bigoplus_{p \in \mathbb{Z}} X_p [p]$$ where $X_p$ is a free $\mathbb{Z}$-resolution of $H_p D$. 
Theorem (Künneth): let $C, D$ be chain complexes of free abelian groups. Then

$$H_n\left(C \otimes \mathbb{Z} D\right) \cong \bigoplus_{k+l=n} H_k(C) \otimes \mathbb{Z} H_l(D) \oplus \bigoplus_{k+l=n-1} \mathbb{Z} H_k(C, H_l(D))$$
Actually, there exist a functional short exact sequence:

\[ 0 \to \oplus H_\ast(C) \otimes H_\ast(D) \to H_\ast(C \otimes D) \to \oplus \text{Tor}_k^R(H_\ast(C), H_\ast(D)) \to 0. \]

\[ k+\ell = n \]

\[ k+\ell = n-1 \]

Carter-Gilbertson:

\[ [x] \otimes [y] \mapsto [x \otimes y] \]

Homological algebra.

Why do I care about the K"unneth theorem?

Because I have the following singular chain complex:

\[ \text{Theorem (Eilenberg-Tate): For spaces } X, Y, \quad C(X \times Y) \cong C(X) \otimes C(Y) \]

natural chain homotopy equivalence.
Example: \[ H_* (S^1 \times S^1) \]

\[
egin{align*}
H_0 (S^1) &= \mathbb{Z} \\
H_1 (S^1) &= \mathbb{Z} \\
H_0 (S^1 \times S^1) &= H_0 (S^1) \otimes H_0 (S^1) = \mathbb{Z} \\
H_1 (S^1 \times S^1) &= H_0 (S^1) \otimes H_1 (S^1) \\
&\quad \oplus H_1 (S^1) \otimes H_0 (S^1) \\
&= \mathbb{Z} \oplus \mathbb{Z} \\
H_2 (S^1 \times S^1) &= H_1 (S^1) \otimes H_1 (S^1) = \mathbb{Z} \\
\end{align*}
\]
Note general commutative rings:

No precise Kunneth theorem
(spectral sequence)

- Over PID's - same as over \( \mathbb{Z} \)
- Over a field - No \( \text{Tor}^1 \) or \( \text{Ext}^1 \)!

Over a general ring, we still have a canonical map:

\[
(H_* C) \otimes_k (H_* D) \rightarrow H_{k+c} (C \otimes_k D)
\]

\(c \otimes [d] \rightarrow [c \otimes d]\)