**CW-complexes and their homology**

**Definition:** A relative CW-complex is a pair $(X, Y)$ (i.e., $Y \subseteq X$) with subspaces called **skeleta**

$Y = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$

$X = \bigcup \limits_{n=0}^{\infty} X_n$

A subset $U \subseteq X$ is open if and only if $U \cap X_n$ is open in $X_n$ for all $n$.
for every \( m = 0, 1, 2, \ldots \), we are given a set \( I_m \) (the set of \( n \)-cells) and a map (called the attaching map)

\[
\varphi_m : I_m \times S^{m-1} \rightarrow X_{m-1}
\]

and we have

\[
X_m = X_{m-1} \sqcup (I_m \times D^m) / x \circlearrowleft S^{m-1}, \text{ } x \in I_m
\]

A CW-pair is of dimension \( \leq n \) if

\[
X_n = X
\]

A CW-complex is a relative CW-complex of the form \((X, \emptyset)\), \((S^{-1} = \emptyset)\)
Examples: \( \mathbb{R}P^n, \mathbb{C}P^n, n \in \mathbb{N} \cup \{\infty\} \).

\( \mathbb{R}P^n = \{ \text{1-dimensional real vector subspaces of } \mathbb{R}^{n+1} \} \)

\( \mathbb{C}P^n = \{ \text{1-dimensional complex subspaces of } \mathbb{C}^{n+1} \} \).

\[ \mathbb{R}^{n+1} \setminus \{0\} \rightarrow \mathbb{R}P^n \quad \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}P^n \]

\[ x \mapsto \langle x \rangle \quad x \mapsto \langle x \rangle \]

The targets have the quotient topology.

\( \mathbb{R}P^n, \mathbb{C}P^n \) are compact.

How \( \mathbb{R}P^n \) a CW complex?

\( \mathbb{R}P^0 \subset \mathbb{R}P^1 \subset \cdots \subset \mathbb{R}P^n \)
\[ \mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^{n+1} \]

\[ (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 0) \]

\[ \mathbb{C}^0 \subset \mathbb{C}^1 \subset \cdots \subset \mathbb{C}^n \]

\[ \mathbb{C}^0 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{n+1} \]

\[ S^{k-1} \xrightarrow{f_k} \mathbb{R}P^{k-1} \]

\[ \Gamma \downarrow \quad \text{Ver.} \quad \mathbb{R}P^k = \mathbb{R}P^{k-1} \bigcup_{s^k} D^k \]

\[ D^k \quad \xrightarrow{\partial^k} \mathbb{R}P^k \]

\[ f_k : S^k \to \mathbb{R}P^{k-1} \]

Think of \( D^k \approx \{ x \in \mathbb{R}^{k+1} \mid \| x \| > 1, x_{k+1} > 0 \} \)

\[ (x_1, \ldots, x_{k+1}) \]
The map \( D^k \to \mathbb{R}P^k \) is \( x \to \langle x \rangle \).

\[ \mathbb{R}P^{k-1} \xrightarrow{f_{k-1}} D^k \to \mathbb{R}P^k \]

is continuous, bijective, and therefore a homeomorphism since the source is compact.

The complex case:

\[ f_{2k} : S^{2k-1} \to \mathbb{C}P^{k-1} \]

\[ \mathbb{C}P^{k-1} \xrightarrow{f_{k-1}} D^{2k} \to \mathbb{C}P^k \]

\[ f_{2k} : x \to <x> \]
Think of $D^k = \{ x \in \mathbb{C}^{k+1} \mid \|x\| = 1, \ x_{k+1} > 0 \}$

$(x_1, \ldots, x_{k+1})$

(in particular, $x_{k+1} \in \mathbb{R}$).

Again:

\[ \mathbb{C}P^{k-1} \cup \mathbb{S}^{2k-1} \xrightarrow{\cup} \mathbb{C}P^k \]

Injective continuous \&

source is compact \& homeo-

\[ \mathbb{C}P^{\infty} = \bigcup \mathbb{C}P^n \]

\[ \mathbb{R}P^{\infty} = \bigcup \mathbb{R}P^n \]

union topology.
The homology of a CW pair

We will construct a "reasonably sized" chain complex $C_*(X,Y)$, and will show it is good enough for computing singular homology.

Recall $\tilde{H}_i(S^n) = \mathbb{Z}$, $n = i$

$\otimes_0$ else.

$\tilde{H}_i(D^n, S^{n-1}) \cong \frac{\tilde{H}_i(D^n)}{\tilde{H}_i(D^n, S^{n-1})}$

Think of $D^n = \bar{D}^n$ as

$\{ (x_1, \ldots, x_{m+1}) \in \mathbb{R}^n \mid x_{m+1} \geq 0 \}$

$D^n = \{ x \in \mathbb{R}^n \mid \| x \| \leq 1 \}$

$x_{m+1} \leq 0$
For a CW pair $(X, X_{n-1})$

\[ H_i(X_n, X_{n-1}) = \tilde{H}_i(X_n/X_{n-1}) = \]

\[ = \tilde{H}_i(V \cup S^n) \]

\[ \text{based on } i \in \text{Im} \]

\[ \bigvee_{i \in I} X_i = \bigcup_{i \in I} / \text{identify base points} \]
\begin{align*}
\text{homotopy} & \quad \tilde{H}_i \left( V, S^m \right) = \mathbb{Z} \quad i = m \\
H_i \left( X_{n-1}, X_{m-1} \right) & \cong H_i \left( X_n, X_{n-1} \right) = H_i \left( X_n \setminus X_{n-1}, X_{n-1} \right)
\end{align*}

\begin{align*}
X_{n-1}^{\varepsilon} &= X_{n-1} \upharpoonright \left( \prod_{i \in I_m} I_i \times S^{n-1} \right) \\
S^{n-1}_{\varepsilon} \subset D^m_{(+)} &\quad \{ x \in D^m_{(+)} \mid 0 < x_{n+1} \leq \varepsilon \} \\
(x_1, \ldots, x_{n+1}) &\quad \bigoplus_{i \in I_m} H_i \left( D^m_{(+)} / S^{n-1}_{\varepsilon} \right) \\
\bigoplus_{i \in I_m} \mathbb{Z} &\quad \bigoplus_{i \in I_m} H_i \left( D^m_{(+)} / S^{n-1}_{\varepsilon} \right) \quad \text{This does not mean direct base product}
\end{align*}
\[ H_i^- (X_n, X_{n-1}) = \mathbb{Z} I_n \quad \text{if} \quad i = n \]

\[ 0 \quad \text{else} \]

Differential: \[ \partial: \mathbb{Z} I_n \to \mathbb{Z} I_{n-1} \]

\[ H_m (X_n, X_{n-1}) \xrightarrow{\partial} H_{m-1} (X_{n-1}) \xrightarrow{\partial} H_{m-2} (X_{n-2}, X_{n-1}) \]

\[ \text{d cell} \]

\[ \text{d cell} \quad \text{d cell} \]

\[ \partial \text{ cell} = 0 \]

\[ H_{n-2} (X_{n-2}) \]

\[ \text{LES} \]

\[ 0 \]

\[ H_{n-2} (X_{n-2}, X_{n-1}) \]
Theorem: There is a canonical isomorphism
\[ H_m(\text{Cell}(X,Y)) \cong H_m(X,Y) \]
for CW-pairs \((X,Y)\).

Example: \( H_k(\mathbb{C}P^m) = \mathbb{Z} \) \( k \) even \( 0 \leq k \leq 2m \)
\[ 0 \quad \text{else.} \]

\[ H_k(\mathbb{R}P^m) = \mathbb{Z} \] There is one cell in every dimension \( 0, 1, 2, \ldots, m \). WHAT IS THE DIFFERENTIAL?
Proof (sketch) of the Theorem:

Lemma 1: If $\dim(X,Y) < k$ then $H_n(X,Y) = 0$ for $m > k$

(Induction on $k$, use Lemma 5).

Lemma 2: If $(X,Y)$ (as pair) has no cells of dimension $< k$ then $H_n(X,Y) = 0$ for $m < k$

(By induction on dimension, $H_n(U \times X_k) = \lim H_n(X_k)$.)

\[ A_1 \overset{f_1}{\rightarrow} A_2 \overset{f_2}{\rightarrow} A_3 \rightarrow \cdots \] abelian groups

$\hat{\lim} A_n = \bigoplus A_n$ / Identify via $f_n$. 

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Note: The proof sketch involves lemmas and mathematical concepts that require a deeper understanding of algebraic topology or a related field. The symbols and terms used suggest a high level of mathematical rigor and detail.
Lemma 3: A homological LES of powers: \( A \subseteq B \subseteq C \) yields

\[
\cdots \rightarrow H_\ast(B, A) \rightarrow H_\ast(C, A) \rightarrow H_\ast(C, B) \rightarrow H_{\ast-1}(B, A) \rightarrow \cdots
\]

Proof: Observe we have a LES

\[
0 \rightarrow C(B, A) \rightarrow C(C, A) \rightarrow C(C, B) \rightarrow 0.
\]

To prove the theorem, one first proves

\[
\otimes \quad H_\ast(X, Y) \cong H_\ast(X_{n+1}, X_{n-2}) \quad \text{ (Use Lemma 4.2 \& LES)}
\]

Pairs involved: \((X, X_{n+1}) \quad (X_{n-2}, Y)\)
\[ H_n(x_i, x_{n+1}) \Rightarrow H_n(x_{n+1}, y) \Rightarrow H_n(x_i, y) \Rightarrow H_{n-1}(x_i, x_{n+1}) \]

\[ = \ldots = 0 \]

\[ H_n(x_{n-1}, x_{n-2}) \Rightarrow H_n(x_{n-1}, x_{n-2}) \Rightarrow H_n(x_{n-1}, x_{n-2}) \Rightarrow H_{n-2}(x_{n-1}, x_{n-2}) \]

\[ = \ldots = 0 \]
$H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial} H_n(X_m, X_{m-2}) \rightarrow H_m(X_{n+1}, X_{n-2})$

$H_m(X_{n+1}, X_{n-2})$

$H_m(X_{n+1}, X_{n-2}) \quad H_m(X_{n+1}, X_n) = 0$

Identifying the differential. Relative vs. absolute

$\xrightarrow{\partial}$

$\xrightarrow{\sim}$

$H_n(D^n, S^{n-1}) \xrightarrow{\sim} H_{n-1}(S^{n-1})$.

$\mathbb{Z}^n \rightarrow \mathbb{Z}^{n-1}$ Think of $\partial$ as a matrix! we want its coefficients
One cell specified by an attaching map

\[ S^{n-1} \rightarrow X_{n-1} \rightarrow X_{n-1}/X_{n-2} = V \rightarrow S^{n-1} \]

What does this map induce in homology?

What does any map \( S^m \rightarrow S^m \) induce in homology?

The degree of a map - warm up for Poincaré duality
Milnor: Topology from a differential viewpoint.

The story is

Theorem (Hopf): Let $M$ be a compact smooth oriented $n$-manifold. Then, orient every tangent space $T_{M_x}$ constant in a small neighborhood. The set of homotopy classes of maps $f: M \to S^n$
Every smooth map $f$ of transverse points in the image is dense (Brown's theorem).

Pick $x \in S^n$ and $x \in f^{-1}(x)$.

$$f^{-1}(x) = \{y_1, \ldots, y_m\}$$

$$\deg(f) = \sum_{i=1}^{m} \text{sign} \left( \det Df_{y_i} \right)$$

For $M = S^n$, a map $f : S^n \to S^n$ has oriented the same degree $k \in \mathbb{Z}$ in sense and forget it and only if

$$k = \text{Hom} f : \text{Hom} S^n \to \text{Hom} S^n \in \mathbb{Z}.$$
(Choose one map of degree $k$.)

Identity $S^n = D^n / S^{n-1}$

$\lambda \neq 0$

$-k$ for $k < 0$

use $\lambda < 0$

$\lambda > 0$, $k > 0$

The example of $\mathbb{RP}^n$. The map involved is

$S^{m-1} \rightarrow \mathbb{RP}^{m-1} \rightarrow D^{m-1}_{(\mathbb{Z})} / S^{m-2}$

$\text{Homothetic image}$

$x \mapsto \lambda x + a$, $x, a \in \mathbb{R}^n$, $\lambda > 0$.

Think $n=2$

$[m-1]$

$\mathbb{RP}^n$ (North pole) $\rightarrow$ {North pole, South pole}

Orientation: $n$ even same, $n$ odd negative
\[ \mathbb{Z} \to \cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \cdots \]

\[ \text{even} \]

\[ \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \]

\[ \text{only if } n \text{ odd} \]

**Proof (sketch) of Hopf's Theorem:**

Must check that degree is well-defined.

Doesn't depend on choice of point, homotopy class of \( f \).

Local independence on \( x \): If \( x \) has a regular point, in some neighborhood, all points are regular.
(inverse function theorem and compactness of \(N\)).

Let \(O_{f,y}\) cannot change sign (local diffeomorphism).

Now it suffices to prove homotopy invariance.

Take a homotopy \(h : f \approx g\) \(h : \mathbb{R} \times [0,1] \to \mathbb{S}^m\). smooth \(\quad\) smooth
approximate \(H \times [0,1]\).

Then choose \(x\) such that \(h^{-1}(x) = \mathbb{1}\)

\(h^{-1}(x)\) is a 1-manifold with boundary.


\(\text{Riemannian metric & parametrization by arc length.}\)
Now to prove that \( \text{deg} : [S^2, S^2] \to \mathbb{Z} \) is onto and injective.

Injectivity: Must show that a map of given degree is homotopic to one of those \( \Theta \).
Lemma: Linear maps of same sign determinant $0 \rightarrow 0$.

$s^n \rightarrow s^n$ are homotopic (through linear maps).

Proof: Elementary row operations $\sim$ $Id$ except sign reversal. $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Arbitrary smooth map

$f: M \rightarrow s^n, \quad f(x) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.
more darts to coordinate neighborhood

linearize on the disks.

Use lemma.

Can't see: Can end up with some +1, -1's. How to cancel?
\[ \sim \quad \sim \quad \sim \quad (\Box) \]

\[ \text{negative } m \times \text{coordinate} = 0. \]