Chain complex of $\mathbb{R}$-modules

\[ \text{Hom}_R(C, \mathbb{R})_m = \text{Hom}_R(C_{m+1}, \mathbb{R}) \]

\[ \text{differential } = \delta \]

\[ f : C_m \to \mathbb{R} \]

\[ \delta f(y) = (-1)^m + 1 f(dy) \]

Chain map
\[ \text{ev} : \text{Hom}_R (C, R) \otimes_R C \rightarrow R \]

\[ \text{chain complex concentrated in dim } 0 \]

\[ 0 \rightarrow 0 \rightarrow R \rightarrow 0 \rightarrow 0 \]

\[ \text{chain map} \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \]

\[ \text{dim } 1 \quad \text{dim } 0 \quad \text{dim } -1 \]

\[ \text{ev}(f \otimes x) = f(x). \]

\[ \text{Lemma: Chain maps of R-modules} \]

\[ f : D \rightarrow \text{Hom}_R (C, R) \]

are in bijective correspondence with chain maps of R-modules

\[ g : D \otimes_R C \rightarrow R \]
Under the correspondence,
\[ f \mapsto g \]
\[ \text{for } \ell = \text{Id} \]
\[ D \otimes_R C \rightarrow \text{Hom}_R(C; R) \otimes_R C \rightarrow R. \]

Proof: If we had just R-modules not chain maps, this is a well known fact:
\[ \text{Hom}_R(A \otimes_R B, C) \cong \text{Hom}_R(A, \text{Hom}_R(B, C)) \]

The correspondence is justified.

To show the chain map structure is preserved both ways,
\[ x \in D_{-m}, \quad y \in C_{n+1} \]
\[ g(x \otimes dy + (-1)^m dx \otimes y) = 0 \quad \text{if and only if} \]
\[ f(x)(dy) + (-1)^m f(dx)(y) = 0 \]
\[ \text{if and only if} \]
\[ f(x)(dy) = (-1)^{m+1} f(dx)(y) \]

\[ (-1)^m (\delta f(x)) y \xrightarrow{\sim} f \quad \text{is a chain map}. \]

We are building a chain map of $R$-modules

\[ \text{Hom}_R(C, R) \otimes_R \text{Hom}_R(D, R) \to \text{Hom}_R(C \otimes_R D, R) \]
By the lemma, it suffices to produce a map

\[ \text{Hom}_R(C_R) \otimes_R \text{Hom}_R(D_R) \otimes_R C \otimes_R D \rightarrow R \]

\[ \downarrow \text{ev} \otimes_R \text{ev} \]

\[ \left( \text{Hom}_R(C_R) \otimes_R C \right) \otimes_R \left( \text{Hom}_R(D_R) \otimes_R D \right) \]

\[ \dim m \quad \dim n \]

\[ T(x \otimes y) = (-1)^{\dim m \cdot \dim n} y \otimes x \]

Now we really have all the ingredients for the cup product: let \( X \) be a space.

The geometric input: The diagonal map
\[ \Delta : X \to X \times X \]

Apply \( C \) (categorical closure): compare with Eilenberg-MacLane:

\[ CX \to C(X \times X) \cong C(X) \otimes C(X) \]

Let \( R \) be a commutative ring, apply \( \text{Hom}_R(\cdot, R) \):

\[ \text{Hom}_R(CX \otimes CX, R) = \text{Hom}(CX \otimes CX, R) \to \text{Hom}(CCX \otimes R, R) \]

\[ \text{Hom}_R(CX \otimes CX, R) \cong (\otimes R) \circ R \]

\[ \text{Hom}_R(CX, R) \cong \text{Hom}(C, R) \]
So we get a natural chain map

\[(+) \quad C^*(X; R) \otimes_R C^*(X; R) \to C^*(X; R)\]

Recall also the canonical homomorphism

\[(\star) \quad \mathcal{H}_k C \otimes_R \mathcal{H}_l D \to \mathcal{H}_{k+l} (C \otimes D)\]

\[
\begin{bmatrix}
[z] \\
[\tau]
\end{bmatrix} \otimes
\begin{bmatrix}
[t] \\
[\delta]
\end{bmatrix}
\mapsto
\begin{bmatrix}
[z \circ \delta] \\
[\tau \circ t]
\end{bmatrix}
\]

From (\(+)\), using (\(\star\))

\[v : \mathcal{H}^k(X; R) \otimes \mathcal{H}^l(X; R) \to \mathcal{H}^{k+l}(X; R)\]

This is called the cup product.

Facts: For cohomology classes \(x \in \mathcal{H}^k(X; R), y \in \mathcal{H}^l(X; R)\)

\[x \circ y \in \mathcal{H}^{k+l}(X; R)\]
\[(xuy)uz = x(u(yuz)) \quad \text{graded commutative (physics: super-commutative)}\]

\[xuy = (-1)^{k \cdot l} yux \leq (\text{because of the behavior of } T)\]

1 \in H^0(X; \mathbb{R}) is defined by \(1(x) = 1\).

\[\Delta^0 = x \quad x \in X\]

\[\Delta^0 \rightarrow X\]

\[= \text{a point } x \in X\]

\[1ux = xul = x.\]

A little bit of algebra: graded objects.

We have a graded \(\mathbb{R}\)-module:

What does this mean?
First point of view: A sequence of $K$-modules 
$(M_n)_{n \in \mathbb{Z}}$

Second point of view: $\bigoplus_{n \in \mathbb{Z}} M_n = M$ (an $K$-module)

but we remember the submodules $M_n \subseteq M$

homogeneous elements of degree $n$.  

Example: (Proof later) The infinite projective
space $\mathbb{R}P^\infty$. Let $R = \mathbb{Z}/2 = \mathbb{F}_2$. 

$H^k(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{Z}/2$ for all $k \geq 0$.

$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{F}_2[x]$ \hspace{1cm} \text{as a ring, } \dim x = 1.

$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$ with respect to $x$ \hspace{1cm} \text{graded cohomologically}

One can then deduce

$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) = \mathbb{F}_2[x]/(x^{m+1}).$

Also, for $C^\infty$

$H^*(C^\infty; \mathbb{Z}) = \mathbb{Z}[y]$ \hspace{1cm} \dim y = 2.
We also need the cap product, which gives $H^*_C(X; R)$ the structure of a graded module over $H^*_C(X; R)$.

\[ \cap : H^b(X; R) \otimes_R H_{q-p}(X; R) \rightarrow H_{q-p}(X; R). \]

No more prerequisites needed! Yeah!

It suffices to construct a chain map (cochains in degrees with - sign)
\[ \cap: C^*(X; R) \otimes_{\mathbb{R}} C_+(X; R) \to C_+(X; R) \]

\[ C^*(X; R) \otimes_{\mathbb{R}} C_+(X; R) \xrightarrow{\text{Id} \otimes_{\mathbb{R}} C_+} C^*(X; R) \otimes_{\mathbb{R}} C_+(X \times X; R) \]

\[ \downarrow \quad \text{Id} \otimes_{\mathbb{R}} \eta \]

\[ C^*(X; R) \otimes_{\mathbb{R}} C_+(X; R) \otimes_{\mathbb{R}} C_+(X; R) \]

\[ \downarrow \quad \text{ev} \otimes_{\mathbb{R}} \text{Id} \]

\[ C_+(X; R) \]

\[ \text{Fact: } x \cap (y \cap z) = (x \cup y) \cap z, \quad x \cup y \in H^\ast(X; R), \quad z \in H_\ast(X; R). \]
In May book, a sign $({}^2\cdot)$ - a different sign convention.

Poincaré duality for topological manifolds.

Let $M$ be an $n$-manifold: every point $x \in M$ has an open neighborhood $U_x$ and $f$ a homeomorphism $h_x : U_x \to \mathbb{R}^n$. Let $R$ be a commutative ring.

\[
 x \mapsto 0 \\
 H_n(\chi; R) : H_n(U_x, U_x \setminus \{x\}; R) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; R)
\]

$R$ isomorphic to $R$, not canonically as an $R$-module.
R is a free \( R \)-module on one generator (e.g., \( 1 \)). Which elements of \( R \) are free generators of \( R \) as a free \( R \)-module? Answer: units.

An orientation of \( M \) is a collection of choices of free \( R \)-module (single) generators

\[
\{ x \in M \mid (u_x, u_x \setminus \{ x \}; R) \ (\cong R) \}
\]

non-canonically, on an \( R \)-module.

with a compatibility condition: when \( y \in \mathcal{U}_x \)

\[
\mathcal{U}_x \cong (\cong R^n)
\]

image \( B' \leftrightarrow B \) ball contains both \( x, y \).
Observation: Every \( n \)-manifold has a unique \( F_2 \)-orientation. (because \( F_2 \) only has one unit)
Proposition: Let \( M \) be a smooth \( n \)-manifold. Then a \( \mathbb{Z} \)-orientation is the same thing as an orientation in the sense of differential topology.

Observation: If \( R \to S \) is a homomorphism of commutative rings, then an \( R \)-orientation on \( M \) gives rise to an \( S \)-orientation (functional of homology with respect to change of coefficients).

\( \mathbb{Z} \)-orientation is universal.

HW: Is there a commutative ring \( R \) such that not every \( n \)-manifold is \( R \)-orientable, but
there exists an $m$-manifold which is $\mathbb{R}$-orientable and not $\mathbb{Z}$-orientable?

(I don't know the answer)

**Proof of the Proposition**: A linear isomorphism $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ induces an isomorphism

$$\text{Hom}(\mathbb{R}^m, \mathbb{R}^n_{\text{odd}}) \to \text{Hom}(\mathbb{R}^n, \mathbb{R}^m_{\text{odd}})$$

**Lemma**: $\text{Hom} \varphi = 1$ if $\det \varphi > 0$

$= -1$ if $\det \varphi < 0$.

(Recall from the degree of a map that $\varphi$ is...
In the smooth case, approximate linearly...

Theorem (the Poincaré duality theorem): Let $R$ be a commutative ring and let $M$ be a compact $R$-oriented $n$-manifold. Then there exists a unique class $c_\omega \in H^n(M; R)$ and that $\forall \omega \in H^n(M; R)$

\[ H_n(M; R) \rightarrow H_n(M, M \setminus \{x\}; R) \xrightarrow{\partial} H_{n-1}(U_x, U_x \setminus \{x\}; R) \]
and the map

\[ H^q(M; \mathbb{R}) \cong H_{n-q}(M; \mathbb{R}) \]

given by

\[ x \mapsto x \cdot \mu \]

is an isomorphism for all \( q \in \mathbb{Z} \).

**Example:** \( H_2(\mathbb{R}; \mathbb{R}) \cong H^0(\mathbb{R}; \mathbb{R}) = \mathbb{R} \{ \text{connected component of } \mathbb{R} \} \).

\[ \text{free } \mathbb{R} \text{-module.} \]

**Example:** Let \( R = \mathbb{F} \) be a field. We already know:

\[ H^2(M; \mathbb{F}) \cong \text{Hom}_\mathbb{F}(H_2(M; \mathbb{F}), \mathbb{F}) = H_2(M; \mathbb{F})^* \]
But by Poincaré duality, \( H^q(M; \mathbb{F}) \cong H_{n-q}(M; \mathbb{F}) \).

So if we know that \( H^1(M; \mathbb{F}) \) have finite rank (e.g. for smooth manifolds) then rank \( H^2(M; \mathbb{F}) = \) rank \( H_{n-2}(M; \mathbb{F}) \).

More function \( \Pi \rightarrow \mathbb{R} \)

(= non-degenerate critical points in the sense of Calc 3).

\( \Pi \) gives a CW decomposition (the handle body decomposition) finitely many cells, otherwise not compact.

Application: Recovering the multiplicative structure.
of $H^*(\mathbb{RP}^n; \mathbb{F}_2)$, $H^*(\mathbb{CP}^n; \mathbb{Z})$. (beginning of this lecture)

Let's do $\mathbb{RP}^n$. We are supposed to get

$$H^*(\mathbb{RP}^n; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{n+1})$$

where $x = 1$.

We do it by induction on $n$. $\mathbb{RP}^1 = S^1$. Holds trivially.

Suppose it is true for $n$. $\mathbb{RP}^n \subset \mathbb{RP}^{n+1}$

$$H^*(\mathbb{RP}^{n+1}; \mathbb{F}_2) \rightarrow H^*(\mathbb{RP}^n; \mathbb{F}_2)$$

1 \rightarrow 1 \rightarrow \quad \text{cell functorial with}

x \mapsto x \quad \text{expect to cell maps}

x^n \mapsto x^n \quad f(x_k) \in Y_k \quad f(x \mapsto y)
\[
\zeta \to 0
\]

generated \( \mathcal{H}^{n+1}(1, \mathbb{R}^{n+1}; \mathbb{F}_2) = \mathbb{F}_2 \)

The only question: Is \( x^n \cup x \neq 0 \in \mathcal{H}^{n+1}(1, \mathbb{R}^{n+1}; \mathbb{F}_2) \)

But I know:

\[
(x^n \cup x) \wedge \eta = \text{orientation class} \cap \text{generator of } \mathcal{H}^{n+1}(1, \mathbb{R}^{n+1}; \mathbb{F}_2)
\]

\[
= x^n \wedge (x \wedge \eta) \neq 0 \text{ by naturality.}
\]

Also note:

\[
\langle x^n, x \wedge \eta \rangle \neq 0
\]

\[
\Rightarrow \mathbb{F}_2
\]

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\mathbb{F}_2
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Comment: The Kشورian product on space 

\[ \langle \cdot, \cdot \rangle : H^k(X; \mathbb{R}) \otimes _\mathbb{R} H^*_\nu(X; \mathbb{R}) \rightarrow \mathbb{R} \]

\[ C^k(X; \mathbb{R}) \otimes _\mathbb{R} C^*(X; \mathbb{R}) \]

\[ \text{Hom}_\mathbb{R}(C(X; \mathbb{R}), \mathbb{R}) \otimes _\mathbb{R} C(X; \mathbb{R}) \rightarrow \mathbb{R} \]

\[ \text{Id} : X \rightarrow X \times X \rightarrow X \]

\[ \sum a_i x_i : 1 \rightarrow \mathbb{C} \]

\[ a_i \in \mathbb{C} \]

\[ x_i \in X \]
When \( R \) is a field, \( H^k(X; \mathbb{F}) \) is isomorphic to \( \text{Hom}_R \left( \mathbb{F}, H_\ast(X; \mathbb{F}) \right) \). Then there is a natural morphism \( \langle \cdot, \cdot \rangle : H^k(X; \mathbb{F}) \otimes H_\ast(X; \mathbb{F}) \to \mathbb{F} \).

**Next time:** Seriously start the proof of Poincaré duality.