Introductory algebraic topology

Some more category theory

What is a diagram?

Example:

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow k & & \downarrow g \\
Z & \rightarrow & T
\end{array}
\]

In a general category \( C \), \( \text{Obj} \), \( \text{Mor} \) \( C \)

are classes. We say \( C \) is small if \( \text{Obj} \), \( \text{Mor} \) \( C \)

are sets.
A diagram in a category $\mathcal{C}$ is a functor
\[ D : I \to C \]
where $I$ is a small category.

A cone on a small category $I$ is a category $\mathcal{C}_I$ where $\text{Obj} \, \mathcal{C}_I = \text{Obj} \, I \cup \{ * \}$

For $x, y \in \text{Obj} \, I$ there is precisely one morphism
\[ \mathcal{C}_I (x, y) = I (x, y) \]
\[ * \rightarrow x \]
for every $x \in \text{Obj} \, \mathcal{C}_I$. 

\[ \text{Diagram} \]
A cone on a diagram \( D : I \to C \) is a diagram \( \overline{D} : CI \to C \) such that \( \overline{D} / I = D \).

\[ (I \subseteq CI) \]

The limit of a diagram is the universal cone.

\[ \begin{array}{ccc}
I & \xrightarrow{D} & C \\
\downarrow c & & \downarrow C \\
CI & \xrightarrow{D_0} & C \\
\end{array} \]

Universality means: For every cone \( \overline{D} : CI \to C \) there exists a unique natural transformation \( \overline{D} \to D_0 \) which is the identity on \( I \).
We write \( \lim D = \lim D = D_0 \). The limit of \( D \)

We are interested in the opposite (dual) concepts (when we turn around arrows)

\[
\text{co-cone } C^0 I = (C(I^0))^{0p}
\]

on a small category

\[
\text{co-cone on a diagram } \quad I \rightarrow^D C
\]
The colimit of $D$ is a (co-)universal co-cone
\[ D^0 : \text{C}^0 \text{h} \text{I} \to D \]

For every co-cone $\overline{D} : \text{C}^0 \text{h} \text{I} \to D$ there exists a unique natural transformation $D^0 \to \overline{D}$ which is the identity on $\text{I}$. 

\[
\text{colim } D = \lim D = D^0(\cdot).
\]
Examples: Limits

\text{Set} \quad \text{(sets, maps)}

\begin{align*}
\text{projections} & \quad X \times Y \\
\text{The product} & \quad X, Y \text{ sets} \\
\prod_{i \in I} X_i & = \lim_{\longrightarrow} \left( X_i \right)_{i \in I} \\
\end{align*}

In any category, \( X \times Y = \lim_{\longrightarrow} \left( X \times Y \right) \)

\[ \prod_{i \in I} X_i \] no morphisms except identities
\[ X \times_2 Y = \{ (x, y) \in X \times Y \mid f(x) = g(y) \} \]

In a general category,

\[ X \times_2 Y \rightarrow X \]

The pullback

\[ Y \rightarrow Z \]

(under product)

Remark: Limits and colimits may or may not exist, but if they do, they are unique (up to isomorphism). Example:

\[ \text{Uniqueness: } s \cong s' \]
\[ X \xrightarrow{f} Y \quad \text{in } \mathcal{V} \]

\[ \lim_{\to} \{ x \in X \mid f(x) = g(x) \} \]

This limit is called the equaliser.

\[ \text{Equal}(f, g) = \lim_{\to} (f, g) \]

\[ \text{Kernel.} \]

\[ \ldots \xrightarrow{f_5} X_5 \xrightarrow{f_4} X_4 \xrightarrow{f_3} X_3 \xrightarrow{f_2} X_2 \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 \]

\[ \lim_{\to} X_n = \{ x = (x_n)_n \in \prod X_n \mid \forall m \geq n, f_m(x_n) = x_{n-1} \}. \]
Universal algebra: a set of operations $m_i$ - any $m_i = 0, 1, 2, \ldots$
& homomorphisms $n_i = 1, \ldots, k$
Example: [Group] 2-ary operation $\circ$
0-ary operation $e$ (neutral element)
1-ary operation $(\cdot)^{-1}$
Equation: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
$a \cdot e = e \cdot a = a$
$a \cdot (a^{-1}) = (a^{-1}) \cdot a = e.$
vector space
(fixed field) [Ring], [Field], [commutative ring]
not an example: \[ \text{field } \iff \forall a \neq 0 \exists a^{-1} \ a(a^{-1}) = 1 \]

In universal algebra, all the limits look the same as in pt.

Example: The product in groups is the cartesian product

\[ \begin{array}{c}
K \\ \downarrow \\
C \times H \\
\downarrow \\
H \\
\end{array} \quad \begin{array}{c}
\text{hom. } x \\
\downarrow \\
\text{morph. } \\
\downarrow \\
\mathbb{Z} \\
\end{array} \quad \begin{array}{c}
\{ (g, h) \in C \times H \mid \alpha(g) = \beta(h) \} \\
\end{array} \]

Same thing with equaliser, same thing with inverse limit
Example: \( \lim \left( \ldots \rightarrow \mathbb{Z}/p^n \rightarrow \mathbb{Z}/p^{n+1} \rightarrow \cdots \right) \rightarrow \mathbb{Z}/p \) \( \rightarrow \mathbb{Q}/p \) \( \rightarrow \mathbb{Z}_p \) \( \rightarrow \mathbb{Q}_p \)

\( \mathbb{Z}_p \) \( p \)-adic numbers.

Example of colimits:

\[ \begin{array}{ccc}
X & \to & \mathbb{Z} \\
\downarrow & & \downarrow \\
Y & \to & T
\end{array} \]

\( X \cup Y \) disjoint union coproduct

\[ K \xrightarrow{f} X \quad \text{in any category, these pushouts} \quad \xrightarrow{g} Y \]

\( \xrightarrow{g} \) colimit \( X \amalg T / \sim \) smallest equivalence relation \( f(k) \sim g(k) \) for all \( k \in K \).
\text{column}(X \xrightarrow{t} \cdots \xrightarrow{g} Y) = \text{in ret : coequaliser}

1 / \sim

\text{column}(X \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} X_n \xrightarrow{f_{n+1}} \cdots) = \prod_{n=1}^\infty X_n / \sim_{\sim_{x \sim f_n(x)}} \text{ for all } x \in X_n.

Groups:

\begin{align*}
G & \xrightarrow{y} G * H \xrightarrow{!} K \\
H & \xrightarrow{\circ} G * H \\
\end{align*}

free product of groups
Free group on a set $S$

$$F(S) = \{a_1^{e_1} \cdots a_n^{e_n} \mid a_i \in S, e_i \in \mathbb{Z} \}$$

words

$$w_1 \cdot a \cdot w_2$$

In each equivalence class:

unique minimal word

$$w_1 \cdot a \cdot w_2 \sim w_1 \cdot a^{k+l} \cdot w_2$$

minimal

For every map $f : S \to H$ where $H$ is a group, there exist a unique homomorphism of groups $\tilde{f} : F(S) \to H$ such that $\tilde{f}(s) = f(s)$ for $s \in S$.

$S \subseteq F(S)$
Applications: \( G \times H = F(G \amalg H) /_{g_1 \ast g_2 \sim g} \)

when \( g_1 g_2 = g \) in \( G \)
\( h_1 h_2 = h \)

This makes \( G \to G \times K \)
\( H \to G \times H \)

inclusions (for commuting)

use universal property and homomorphism theorem

The product \( K \to H \)

of groups \( G \to H \times K \to H \times G / \sim \)
\( f(k) \sim \varrho(k) \) for \( k \in \mathbb{K} \).

Let \( \text{equivalences} : \quad C \xrightarrow{\sim} N \)

\( H/\sim \) smallest congruence

\( f(g) \sim \varrho(g) \) for all \( g \in \mathbb{G} \).

When \( (X_1 \to X_2 \to X_3 \to \ldots) \) a fresh fact

"same as in sets."

Abelian groups \( A \parallel B = A \oplus B \) same as product

\((\text{Caution: } \prod A_i \neq \bigoplus A_i)\)
\[ A \oplus B \to C \]
\[ \text{abelian group} \]
\[ f \circ g \]
\[ f \circ g (a, b) = f(a) + g(b). \]

\[ f : A \to C \]
\[ g : B \to C \]

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Why are limits on universal algebras

"the same" as in set, but relations are not?

**Hint:** The free group is a big clue!

In general, we have a concept of adjoint functors.
let $C, D$ be categories, $F : C \to D$, $G : D \to C$ be functors. We say that $F$ is \underline{left adjoint to $G$} (or $G$ is \underline{right adjoint to $F$}) if there is a natural bijection

\[
C(X, GY) \cong D(FX, Y)
\]

in $\text{set}$.

Example: I claim the free group functor

$F : \text{set} \to \text{Grp}$ \quad \text{category of groups}

is left adjoint to the forgetful functor $U : \text{Grp} \to \text{set}$.
\[ \mathcal{C} \cup (F, S, H) = \text{Set}(S, UH) \]

For any type of universal algebra, there is a free algebra functor \( \{ \text{free monoid}, \text{free algebra over } S, \text{free ring}, \text{free comm. ring}, \text{free module over } R \ldots \} \) left adjoints to the forgetful functors, not a free field (does not exist).

The punchline: A left adjoint preserves colimits, a right adjoint preserves limits.

Example: forgetful functor on universal algebras.
Proof of "the punchline": let's just prove that a right adjoint preserves limits.

$\Gamma : \mathcal{E}_1 \to \mathcal{E}_2$ left adjoint

$\mathcal{E}_2 \to \mathcal{E}_1$ right adjoint

$\mathcal{E}_1 (X, G(T)) \cong \mathcal{E}_2 (F(X), Y)$
maps given by the adjunction

\[ FT \to \mathbb{E} \ni \text{ is the limit} \]

\[ FT \to \mathbb{E} \ni \text{ is equivalent to } T \to \mathbb{E} \]

By naturality,

\[ \mathbb{E} \ni \text{ is the limit of this diagram.} \]
This is quite important in homotopy theory.

\[
\text{Top} = \text{Topological spaces, continuous maps}
\]

"look like" limit of sets

\[
\text{Top has all limits and colimits.}
\]

\[
\prod_{i \in I} X_i
\]

- Cartesian product

- basis of topology \( \prod_{i \in I} U_i \)

where \( \mathcal{F} \subseteq I \) finite,

\[
U_i = X_i \text{ when } i \notin \mathcal{F}.
\]
A set limit of a diagram is a subset of the product of the object in the diagram.

The topological spaces limit is the same set with the induced topology.

To understand colimits of topological spaces, we understand the disjoint union:

\[
\coprod_{i \in I} X_i \quad U \subset \coprod_{i \in I} X_i \text{ open if and only if each } U \cap X_i \text{ is open in } X_i.
\]

All set colimits are quotient of \( \coprod_{i \in I} X_i \) in the diagram.
by equivalence relations. The corresponding topological colimit is the same set with the quotient topology.

\[ X \xrightarrow{f} Y \]

The quotient topology on \( Y \) : \( U \subseteq Y \) open if and only if \( f^{-1}(U) \) open.

\[ \therefore \text{ Topological spaces have all limits and colimits.} \]

There is a construction very important in sets: the exponential adjunction.
\( X, Y, Z \) sets,

\[ \text{Set}(X, \text{Set}(Y, Z)) \cong \text{Set}(X \times Y, Z) \] naturally.

Is there a functor \( \text{Map}(\_ , \_ ) : \text{Top}^0 \times \text{Top} \to \text{Top} \) such that

\[ \text{Top}(X, \text{Map}(Y, Z)) \cong \text{Top}(X \times Y, Z) \]?

Answer: NO.

Because of this problem, \( X \times Y \) does not preserve colimits. This is why a product of two CW complexes (in Top) needn't be a CW complex.
Solution (Steenrod): A convenient category of topological spaces (Michigan Math. Journal)

For the purposes of algebraic topology we will henceforth change the category of spaces...

Steenrod's idea: Recall that a compact space is a topological space $X$ such that every open cover $(U_i)$ of $X$ has a finite subcover.

$\forall i \in I \exists F \subseteq I$ finite $\cup_{i \in F} U_i = X$

$\bigcup_{i \in I} U_i = X$
A space $X$ is compactly generated if:

A set $U \subseteq X$ is open if and only if for every $Z \subseteq X$ compact, $Z \cap U$ is open in $Z$.

(Examples: Every metric space is compactly generated.

$\bullet \cdot \cdot \cdot$ $\infty + 1 = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n} \} \subseteq \mathbb{R}$)

$\Rightarrow$ Every manifold is compactly generated.

Every CW complex $\Rightarrow$

every compact space.

Steenrod's category: Spaces $=$ Compactly generated spaces, continuous maps.
Spaces $\subseteq \text{Top}$ is a left adjoint.

Right adjoint $\mathcal{C}$: making a space compactly generated

$X = X$, $U \subseteq X$ is open in $X^c$ by definition

as a set,

$\forall U \cap \tilde{U}$ is open in $\tilde{U}$

for all $\tilde{U} \subseteq X$ compact.

(The key observation: $\tilde{U}$ is compact in $X^c$ if and only if $\tilde{U} \cap \tilde{Z}$ is compact in $\tilde{X}$.)

$X \xrightarrow{\text{Id}} X$ is continuous.
Conclusion: Colimits of compactly generated spaces are compactly generated \((\text{in Top})\). Generated \(\Rightarrow\) CW complexes are compactly generated.

Compactly generated spaces have limit:

"TAKE THE UNIT IN \(\text{Top}\) AND MAKE IT COMPACTLY GENERATED."

Steenrod's Theorem: \(X, Y\) is compactly generated:

\[
\text{Map}(X, Y) = \{\text{continuous maps } f: X \to Y \mid f \text{ is compact-open} \}
\]

Compact-open topology: subsets \(K \subseteq X\) compact \(U \subseteq Y\) open \(f: X \to Y\) if \(f(K) \subseteq U\).
Map \( (X, Y) \) has the compact-open topology made 
compactly generated (apply \((2)\)).

Theorem:

\[
\text{Map} \left( X \prod Y, Z \right) \cong \text{Map} \left( X, \text{Map} \left( Y, Z \right) \right).
\]

in compactly 
generated spaces

Caution: 
\( X \prod Y \subset (X \times Y)^c \) \((\neq X \times Y\)

in general).

We say \( X \) is pseudo-Hausdorff 
if

\[
X \subset (X \times X)^c \text{ is closed.} \quad \text{Hausdorff} \quad \text{if} \quad X \subset X \times X \text{ closed.}
\]

\[
x \mapsto (x, x)
\]