This package is optional reading for Math 110 students. It is highly recommended for students with less prior proof-writing experience.

A Primer on Mathematical Proof

A *proof* is an argument to convince your audience that a mathematical statement is true. It can be a calculation, a verbal argument, or a combination of both. In comparison to computational math problems, proof writing requires greater emphasis on mathematical rigor, organization, and communication.

A typical theorem may have the form:

Theorem. *Under Conditions A, Statement B is true.*

Here, A is called the *hypothesis* or *assumptions*, and B is called the *conclusion*. Note that mathematicians use the word "hypothesis" differently than it is used in the scientific method. For a mathematician, a 'hypothesis' is not an statement with an absolute 'true' or 'false' value, it is a description of a situation in which the conclusion of the theorem holds.

A proof is a series of statements, each following logically from the previous, to reach the conclusion – using only the hypotheses, definitions, and known true statements.

Examples of Theorems and Their Proofs

Theorem 1. *Let p and q be primes, and suppose p divides q. Then $p = q$ or $p = -q$.*

Proof. Since q is prime, by definition, its only divisors are $1, -1, q$, and $-q$. Moreover since p is prime, by definition, it cannot equal 1 or -1. Therefore p must be q or $-q$, as claimed. $\square$

Theorem 2. *Suppose that a and n are positive integers such that a^n is divisible by 2. Then a^n is divisible by 2^n.*

Proof. Positive powers of odd integers are necessarily odd, hence, a^n can only be even if a is even. Therefore we can factor $a = 2b$ for some integer b, and

$$a^n = (2b)^n = 2^n b^n.$$

We conclude that a^n is divisible by 2^n.

$\square$

There is some discretion involved in writing proofs, in deciding which statements are well-known or self-evident and which need further justification. For example, in the above proof, we used the fact that products of odd integers are odd without justification. See Exercise 8.

Common Mistakes in Proof-Writing

Common Mistake #1. Starting with the conclusion, and working backwards toward the hypotheses.

This approach can be a good way to brainstorm how the proof will work – but it is *not* appropriate for the final write-up of the proof. At best, such arguments are poor exposition, and at worst they are logically unsound. Always start a proof with the hypotheses and known truths, and work toward the conclusion.

Common Mistake # 2. Checking the conclusion in a few specific examples, and extrapolating that it always holds.

A proof must give a rigorous argument that the conclusion holds in *all* situations satisfying the hypotheses.

For example, if you want to prove that some property holds for all real numbers, then assign a variable (such as x) to represent an arbitrary real number, and prove that property holds for this arbitrary number x. Because x can represent any real number, you can conclude that the property holds for every real number. It is not enough to simply plug in a few numbers for x and check in those cases. Playing around with specific numbers may help you discover the proof, but it is not sufficient for the final proof write-up.

In contrast, you can *disprove* a statement by finding a single example where the hypotheses hold but the conclusion fails.

Writing Tip: All said, for complicated proofs, it can be useful to the reader to explain an argument in the case of a particular example before giving the argument in general. This is not logically necessary, but it can help the reader understand the general proof.

Common Mistake #3. Proof by (insufficiently general) picture.

Similarly, it is often not enough to draw a picture. For example, if you are proving a result about all functions $\mathbb{R} \to \mathbb{R}$, it is not enough to analyze the graph of a particular function.

Pictures can sometimes be used to give rigorous proofs, but only if you can justify why the picture addresses all possible situations satisfying the hypotheses. In these cases, it is usually necessary to include some full English sentences explaining to the reader how to interpret the picture.

Writing Tip: Even if the picture itself only addresses a special case of your proof, it can still be a powerful visual aid to help your reader understand your reasoning. For example, even if you are giving a proof about all functions $\mathbb{R} \to \mathbb{R}$, you may want to include an annotated graph of a particular function to illustrate your arguments. Having both a general written argument and a picture can help you achieve the dual proof-writing goals of justifying your claims rigorously *and* communicating clearly to you reader.

Common Mistake #4. Assuming the conclusion.

Suppose, for example, you want to prove that two quantities or expressions A and B are equal. It is a common error to start this proof by equating $A = B$ and then simultaneously manipulating both sides of the equation. This is *not* a valid approach to the proof. Do not start a proof that $A = B$ with the statement $A = B$. It is possible to "prove" false statements this way. Instead, choose one side of the equation (A or B) and work through a sequence of equalities until you arrive at the other side.

For example, the following is a proof that $(x - y)^2 + 4xy = (x + y)^2$ for all real numbers x and y:

$$
\begin{aligned}
(x - y)^2 + 4xy &= (x^2 - 2xy + y^2) + 4xy \\
&= x^2 + y^2 - 2xy + 4xy \\
&= x^2 + y^2 + 2xy \\
&= (x + y)^2
\end{aligned}
$$

Common Mistake #5. Making steps in the proof without adequate justification.

For this course, you should give justification for any facts that aren't obvious. If you invoke a result that was proved earlier in the course, then you should cite the result and explicitly verify that its hypotheses are satisfied. As a general guideline, the arguments in your proof should be detailed enough to convince another student in this course. Use your judgment about how much to write.

Be sure, too, that the overall structure of the proof is clear. A series of statements or computations are not a complete proof unless it is explained how they connect and why they imply the final result. Most proofs should include full English sentences. Keep in mind that the main goal of the proof is to *communicate* a mathematical argument to the reader.

Common Mistake #6. Overlooking special cases.

For example, if a step in the proof involves dividing by an unknown real number x, you will need to separately treat the case that $x = 0$.

As another example, many proofs about properties of a set S will require a separate argument for the case that S is the empty set $\varnothing$.

Common Mistake #7. Undeclared variables and undefined notation.

A symbol such as x is meaningless until it is assigned a meaning. A proof should include a declaration of all the objects involved, for example "let x be a real number", "let $n \in \mathbb{Z}$", or "let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$". Similarly, any nonstandard notation or shorthand should be defined. Communicate!

Writing tip: There are many customary notational choices in mathematical writing, for example, x, y, z, w are frequently real numbers; f, g, h, ϕ, ψ are frequently functions; n, m, k are often integers; p, q, ℓ are often prime integers; i, j, k are often indices, etc. It can help the reader to stick to these conventions when possible—and deviating too far from them can cause confusion. Although it would be technically correct to, say, let x denote a function of a real number f, this is inadvisable.

Even though these notational choices are unofficial standards, each symbol must still be explicitly defined.

Common Mistake # 8. Imprecise statements.

An argument cannot be rigorous if it involves ambiguous or poorly defined statements. Be specific. Use mathematical terminology, and use it correctly.

For example,

Imprecise Sentence: "The sine function looks the same every 2π."

Better Sentences:
"The sine function takes on the same value at any two points in its domain that have a difference of 2π."
"For every real number x, $\sin(x) = \sin(x + 2\pi)$."
"The sine function is periodic with period 2π."

Mathematical statements also require quantifiers.

For example,

Imprecise Sentence: "Therefore $f(x)$ vanishes."

What does this mean? Does $f(x)$ vanish at every point x? For at least one point x? At a particular point x? Spell it out!

Common Mistake # 9. Misuse of the equal sign.

Always reserve use of the equal sign "=" for quantities (numbers, functions, sets, elements of a set, etc.) that are exactly equal to each other. It is common to see students improperly use "=" as shorthand for the verb "to be" or to use it between quantities that are not, in fact, equal.

For example,

Improper Sentence: "f = positive"

Better Sentences:
"The function f is a positive function."
"The function f satisfies $f(x) > 0$ for all x in its domain."

For example,

Assigned Problem: Differentiate $f(x) = x^2$.

Improperly written solution:

$$f(x) = x^2$$
$$= 2x.$$

Properly written solution:

$$f(x) = x^2$$
$$f'(x) = 2x.$$

Common Mistake #10. Missing context.

For every mathematical assertion in your proof (for example "the function $f(x)$ is continuous"), you must make it clear to your reader

- whether this is a logical deduction from the previous line of the proof,

- whether this is a fact you believe is self-evident,

- whether this is an application of a previous result in the class, or

- whether this is a claim that you intend prove (but haven't yet).

One way to make this distinction clear is to simply state the context in words. Some useful vocabulary are *since, because, so, thus, therefore, hence*, etc. And, you can use phrases like, "This statement follows directly from the definitions." "Our goal is to prove the following statement." "This statement follows from our previous result, [statement of previous result], which applies because [verification that the hypotheses hold]".

Common Mistake #11. General bad communication.

If you were writing an essay, you would not invent your own words, symbols, or shorthand. You would not write idea fragments without context or support. You would not write a chaotic jumble of ideas across the page with a network of arrows directing your reader in circles.

Please treat your proofs the same way. Remember that the primary purpose of your write-up is to communicate your reasoning to your reader. Please write neatly, grammatically, and in a sensible linear order. The proofs in the textbook (and in class) are good models for proper formatting.

It is good practice to re-read your proofs, imagining how they would look to someone else. Is it clear what all the words and symbols represent? Is it clear what order the proof should be read in? Is it clear which statements are assumptions, which are statements that you are about to prove, and which are deductions? Is the purpose of each line clear – what is being deduced, and why this deduction is logically sound? Does the proof start with known facts, and progress toward the desired conclusion? Or, alternatively, start with one side of a desired equation, and progress through a sequence of equalities toward the other side? Consider having a second person proofread your work, and tell you which aspects need clarification.

Writing tip: For longer proofs, consider modularizing your argument by breaking it down into a series smaller claims, each followed by its own proof. Mathematicians often call minor or preliminary statements **Propositions** or **Lemmas**, major results are **Theorems**, and consequences of the main result are **Corollaries**. (There is no formal distinction between the terms proposition, lemma, theorem, and corollary, and the choice of title for a given statement is subjective.)

Be sure to clearly distinguish the statement of each claim from the body of its proof. For very long arguments, number the claims and reference them by number ("By Proposition 2 we infer ... "). Breaking down a large argument into bite-sized pieces can make it easier for the reader both to read individual pieces, and to grasp the structure of the overarching argument.

Example of a False Statement and Counterexample

To prove that a statement is true, you must show that the conclusion holds in every situation that satisfies the hypotheses. You can, however, prove that a statement is false by finding a single instance where the hypotheses hold but the conclusion fails. This is called a *counterexample* to the statement.

False Statement. *For all real numbers a and b, $|a + b| = |a| + |b|$.*

Counterexample. Consider the example $a = 1$, $b = -1$. Then

$$|a + b| = |0| = 0 \qquad \text{but} \qquad |a| + |b| = 1 + 1 = 2.$$

It is therefore not always true that $|a + b| = |a| + |b|$.

# Logic: Conditional Statements and Equivalence

Implication

Let A and B denote properties or mathematical statements. The following notation means "A implies B":

$$A \implies B.$$

Other ways to write this:

- If A is true, then B is true.

- A is true only if B is true.

- B is true whenever A is true.

- B is true if A is true.

- Everything with property A also has property B.

- The set of things with property A is contained in the set of things with property B.

Venn diagram
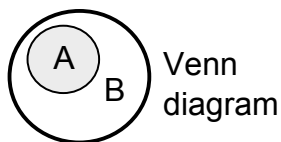
We write $A \not\Longrightarrow B$ to mean "A does not imply B". This means that there is *at least one* example where A holds, but B fails.

Examples of Conditional Statements

"An integer n must be even if it is divisible by 4."

$$\left(n \text{ is divisible by } 4 \right) \Longrightarrow \left(n \text{ is even} \right)$$

"A real number r can be greater than 10 only if it is strictly positive."

$$\left(r > 10 \right) \Longrightarrow \left(r > 0 \right)$$

"The composition $g \circ f$ of two functions f and g is continuous whenever both functions are continuous."

$$\left(f \text{ and } g \text{ are continuous} \right) \Longrightarrow \left(g \circ f \text{ is continuous} \right)$$

"Any differentiable function f is continuous."

$$\left(f \text{ is differentiable} \right) \Longrightarrow \left(f \text{ is continuous} \right)$$

Contrapositive

The conditional statement

$$\text{Property } A \text{ holds } \Longrightarrow \text{ Property } B \text{ holds}$$

is logically equivalent to its *contrapositive*, the statement that

$$\text{Property } B \text{ fails } \Longrightarrow \text{ Property } A \text{ fails}$$

Both of these implications mean precisely that there is no situation where A holds and B fails.

Example of a Statement and its Contrapositive

For example, the statement

   "If n is an integer, then n is a rational number"

has precisely the same meaning as the statement

   "If n is not a rational number, then n cannot be an integer".

Both say that the set of integers is contained in the set of rational numbers.

Converse

The conditional statement

$$A \implies B$$

is *not* equivalent to the statement

$$B \implies A,$$

which is called its *converse*.

Example of a True Implication with a False Converse

Let N be an integer. It is true that:

$$\left(N \text{ is divisible by } 6 \right) \implies \left(N \text{ is divisible by } 3 \right)$$

The converse statement, however, is false.

$$\left(N \text{ is divisible by } 3 \right) \;\not\!\!\!\implies \left(N \text{ is divisible by } 6 \right)$$

Be careful not to confuse a conditional statement with its converse!

Two-Way Implication

We write

$$A \iff B$$

to mean "A is true if and only if B is true". We say that A and B are *equivalent*.

Proving an "if and only if" statement requires two steps: showing $A \implies B$, and showing $B \implies A$.

Examples of Two-Way Implication

Theorem 3. *Let $P(x)$ be a polynomial in x, and let r be a real number. Then $P(r) = 0$ if and only if $(x - r)$ is a factor of P.*

Theorem 4. *Let a and b be real numbers. Then $|a + b| = |a| + |b|$ if and only if $(ab) \geq 0$.*

Theorem 5. *An integer m is divisible by 3 if and only if the sum of its digits is divsible by 3.*

Theorem 6. *An integer p (with $p \neq -1, 0, 1$) is prime if and only if it satisfies the following conditional statement: Whenever p divides a product of integers $a_1 a_2 a_3 \cdots a_n$, p must divide at least one of the factors a_i.*

A Warning About Definitions

There is an unfortunate convention in mathematics to write definitions as "if" statements, even though they should be interpreted as meaning "if and only if".

For example, the statement

"Integers a and b are said to be *coprime* if their greatest common divisor is 1"

should be understood to mean

"Integers a and b are coprime if and only if their greatest common divisor is 1".

The statement

"If an interval $I \subseteq \mathbb{R}$ does not contain its endpoints, it is called *open*."

implicitly means

"An interval $I \subseteq \mathbb{R}$ is open if and only if it does not contain its endpoints".

Quantifiers

The symbol $\forall$ is shorthand for "for all" or "for every". The symbol $\exists$ means "there exists (at least one)". Some authors use the symbol $\exists!$ to indicate existence and uniqueness; it is read "there exists exactly one".

For example, the sentence

$$\forall \, n \in \mathbb{Z} \text{ such that } n \geq 2, \; \exists \, p \in \mathbb{Z} \text{ such that } p \text{ is prime and } p | n$$

would be read

"For every integer $n \geq 2$, there exists an integer p such that p is prime and n is divisible by p."

This shorthand can be convenient for writing notes, but is generally considered inappropriate in formal proofs and expository math writing. It is typically not used in published mathematics papers. In this course we ask that you avoid these symbols in long-form homework solutions and in your WIM project.

The *order of quantifiers* can make a critical difference in the meaning of a mathematical sentence. For example, of the statements below, Statement (A) is true, and Statement (B) is false.

(A) $\forall \, n \in \mathbb{Z} \, \exists \, m \in \mathbb{Z} \text{ such that } m > n$

(B) $\exists \, m \in \mathbb{Z} \text{ such that } m > n \, \forall \, n \in \mathbb{Z}$

In Statement (A), the number n is fixed first, and the choice of m may depend on n. The statement is true since, given any particular integer n, it is indeed possible to find a larger integer m. In contrast, Statement (B) asserts that there is an integer m larger than all other integers. Since the variable m is specified first, we must make a single, global choice of a number m, independent of n, which is greater than every integer n.

This example demonstrates how the order of quantifiers determines how the variables may (or may not) depend on each other. Some authors may rewrite Statement (A) to emphasize this dependence, as follows:

(A) rephrased $\forall \, n \in \mathbb{Z} \, \exists \, m = m(n) \in \mathbb{Z} \text{ such that } m > n$

(A) rephrased $\forall \, n \in \mathbb{Z} \, \exists \, m = m_n \in \mathbb{Z} \text{ such that } m > n$

The expressions $m = m(n)$ and $m = m_n$ mean "m is a function of n", that is, a different value of m may be chosen for each fixed n. Both of these restatements of (A) are logically equivalent to Statement (A) above, but may be easier for the reader to parse.

This subtlety concerning order of quantifiers features, for example, in the formal definition of the limit.

Inclusive "or"

By convention, mathematicians use the word "or" to be *inclusive*, that is, the statement "A or B" means either A is true, B is true, or both A and B are true.

For example, the statement

"Every real number is nonnegative or nonpositive."

asserts that every real number x is nonnegative, nonpositive, or both (as is the case with $x = 0$).

The statement:

"Let a, b, p be integers such that p is prime and ab is divisible by p. Then p divides a or b."

asserts that at least one of a and b is divisible by p, and they may both be divisible by p.

An *exclusive 'or'* must be written out explicitly, for example:

"Every integer is either even or odd, but not both."

A second example:

"Let a, b, p be integers such that p is prime and ab is divisible by p but not p^2. Then p divides exactly one of a and b."

Negation

The *negation* of a statement is its logical opposite, sometimes also called the *logical complement*. That is, a statement A is true precisely if its negation is false, and vice versa.

The negation of the statement "A and B are true" is "A or B is false". Remember that "A or B is false" this means that at least one of A or B is false, and both may be false.

The negation of "A or B is true" is "A and B are both false".

The negation of a statement of the form "For all x, proposition $P(x)$ holds" has the form "There exists an x such that proposition $P(x)$ fails."

The negation of the statement "There exists an x such that proposition $P(x)$ holds" is "For every x, proposition $P(x)$ does not hold".

An example:

(C)   For all integers m there exists an integer n which is not divisible by m.

Negation of (C)   There exists an integer m such that every integer n is divisible by m.

Proof Techniques

Below are some standard methods of proof.

Technique #1: Direct Proof

Begin by stating the hypotheses. Use known facts, definitions, calculations, and reasoned deductions to make a sequence of logical inferences that culminates with thte conclusion.

Example of a Direct Proof

Theorem 7. *Let $a, b, c \in \mathbb{Z}$. If $a|b$ and $b|c$ then $a|c$.*

Proof. Suppose that a, b, c are inetgers such that $a|b$ and $b|c$. The statement that $a|b$ means, by definition, that there exists some integer b' so that $b = ab'$. Similarly, the statemen $b|c$ means $c = c'b$ for some inetger c'. But then, if we substitute the fisrt equation into the second we obtain

$$c = c'b$$
$$c = c'(ab')$$
$$c = a(c'b').$$

Thus, by definition, c is divisible by a. ☐

Technique #2: Proof by Contradiction

Suppose that the hypotheses are true, but that the conclusion is false. Reach a contradiction. Deduce that if the hypotheses are true, the conclusion must be true too.

Examples of Proofs by Contradiction

Theorem 8. *The real number $\sqrt{2}$ is irrational.*

Proof. We proceed by contradiction. Suppose that $\sqrt{2}$ were a rational number; this means that we can write $\sqrt{2} = \dfrac{m}{n}$ as a reduced fraction, with coprime integers m and n. Squaring both sides of this equality and rearranging gives:

$$2 = \frac{m^2}{n^2}$$
$$m^2 = 2n^2$$

Thus m^2 is divisible by 2. By Theorem 2, m^2 must in fact be divisible by 4, which implies that n must also be even. But then 2 is a common divisor of m and n, which contradicts our premise that $\dfrac{m}{n}$ was a reduced fraction. We conclude that $\sqrt{2}$ cannot be rational.
☐

Theorem 9. *If $|x| < \epsilon$ for every real number $\epsilon > 0$, then $x = 0$.*

Proof. Suppose for the sake of eventual contradiction that $|x| < \epsilon$ for every positive number ϵ, but $x \neq 0$.

Since $x \neq 0$, necessarily $\dfrac{|x|}{2} > 0$, so in particular $|x| < \epsilon$ for the positive number $\epsilon = \dfrac{|x|}{2} > 0$. This means

$$|x| < \frac{|x|}{2}.$$

But, $|x| \neq 0$ by assumption, so we can divide both sides by $|x|$ to conclude that $1 < \frac{1}{2}$, a contradiction!

Thus, if $|x| < \epsilon$ for every real number $\epsilon > 0$, it must be the case that $x = 0$. ☐

Technique #3: Proof by Exhaustion

This involves breaking down the proof into a number of cases. Argue that any instance of the theorem falls into one of the cases, and show that the theorem is true in every case.

Example of a Proof by Exhaustion

Theorem 10. *For any real number a, $|a|^2 = a^2$.*

Proof. Since a must satisfy either $a \geq 0$ or $a < 0$, it suffices to prove the result for these two cases.

If $a \geq 0$, then $|a| = a$, so

$$|a|^2 = a^2.$$

If $a < 0$, then $|a| = -a$, so

$$|a|^2 = (-a)^2 = (-1)^2 a^2 = a^2.$$

In all cases, $|a|^2 = a^2$. ☐

Technique #4: Proof by Induction

This is a technique used to prove a mathematical proposition $P(n)$ involving a natural number n is true for every value of n. $P(n)$ might be a formula

$$\text{for example,} \qquad 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

or a statement involving a variable n

for example,   "For every natural number n, $n^3 + 2n$ is divisible by 3."

A proof by induction is structured as follows:

1. **The Base Case.** Prove that the proposition holds when $n = 1$

2. **The Inductive Step.** Prove that, under the assumption that proposition holds for $n = k$ for some natural number k, it follows that it holds for the subsequent number $n = k + 1$. The assumption that $P(k)$ holds is called the *inductive hypothesis*.

   The inductive step involves proving a *conditional* statement:

   If $P(n)$ holds when $n = k$, then $P(n)$ holds when $n = k + 1$.

3. **Conclusion.** Conclude that the proposition holds for every natural number n.

   Because the proposition holds when $n = 1$, the inductive step implies that it must hold when $n = 2$, and therefore it must hold when $n = 3$, and when $n = 4$, and so forth, for every natural number n.

The reasoning behind a proof by induction is often compared to 'the domino effect'.

Example of a Proof by Induction

Theorem 11. *For any integer $n \geq 1$,*

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

Proof. We begin the base case, when $n = 1$. In this case, the left hand side of the equation has a single term,

$$2n - 1 = 2(1) - 1 = 1,$$

and the right-hand side of the equation is

$$n^2 = 1^2 = 1.$$

These qauntities agree, so equality holds when $n = 1$.

Next we make the inductive step. Let k be a fixed positive integer, and assume the formula holds when $n = k$. Specifically, we assume that

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2 \qquad \text{(Inductive hypothesis)}.$$

Our goal is to show that, under this assumption, the formula holds when $n = (k + 1)$. Specifically we want to show that:

$$1 + 3 + 5 + \cdots + \big(2(k + 1) - 1\big) = (k + 1)^2.$$

We expand left-hand side:

$$1 + 3 + 5 + \cdots + (2k - 1) + \big(2(k + 1) - 1\big)$$
$$= \Big(1 + 3 + 5 + \cdots + (2k - 1)\Big) + \big(2(k + 1) - 1\big)$$

The quantity $\Big(1 + 3 + 5 + \cdots + (2k - 1)\Big)$ is, by the inductive hypothesis, equal to k^2, so we have:

$$\Big(1 + 3 + 5 + \cdots + (2k - 1)\Big) + \big(2(k + 1) - 1\big)$$
$$= \Big(k^2\Big) + \big(2(k + 1) - 1\big)$$
$$= k^2 + 2k + 1$$
$$= (k + 1)^2 \qquad \text{as desired,}$$

and so we conclude that (under the inductive hypothesis) the formula holds when $n = (k + 1)$.

Conclusion: Our base case demonstrates that the equation holds when $n = 1$. Thus, the inductive step implies that it must hold when $n = 2$, and therefore it must hold when $n = 3$, and when $n = 4$, etc, and in this way we conclude that the proposition holds for any integer $n \geq 1$.

$\square$

A tip about writing proofs by induction (and, in fact, all proofs): be sure to clearly explain the significance of any formulas you write down. You must distinguish for the reader which equations you have assumed to be true, which are statements that you wish to prove, and which equalities you've already concluded hold. Moreover, give an explanation of any non-obvious steps in your computations. It may help to explicitly indicate where you invoke the inductive hypothesis.The proof should include many full English sentences. Remember the goal is not just to write down the computation needed in the inductive step, but to communicate the full logical structure of the proof to the reader.

Suggested Exercises

The following questions are optional, but are recommended for students with less experience in mathematical proof-writing. You're invited to hand in solutions to any of these problems to Jenny for feedback.

Math 110 students will not be tested directly on the material in this handout, but we will assume the class is comfortable with the mathematical conventions and the principles of proof-writing described here.

1. Read through this handout. Make note of any questions you have, or parts of the handout you found confusing.

2. Briefly explain why a mathematical statement of the form "Under conditions A, conclusion B is true" cannot be proven with a single example, but it can be disproven with a single example.

3. Briefly explain why a conditional statement $A \implies B$ is equivalent to its contrapositive.

4. Suppose we know that $A \implies B$ and and we are in a situation where A is false. What can we conclude about B? (Trick question!)

5. Write the following conditional statements in the form $A \implies B$, and write the converses. For each statement, indicate whether the statement is true or false, and indicate whether its converse is true or false.

   (a) If n is an integer, then $2n$ is an integer.

   (b) A number $5n$ is rational only if n is rational.

   (c) Let a and b be real numbers. If $|a| < |b|$, then $a < b$.

6. Write the negation of the following statements.

   (a) The integer p is prime.

   (b) The integers m and n are divisible by 12.

   (c) If $5n$ is an integer then n is an integer.

7. Below are two mathematical statements written in shorthand.

$$\textbf{(A)} \quad \exists\, n \in \mathbb{Z} \text{ such that } \forall\, p \in \mathbb{Z} \text{ with } p \text{ prime}, p|n$$

$$\textbf{(B)} \quad \forall\, p \in \mathbb{Z} \text{ with } p \text{ prime}, \exists\, n \in \mathbb{Z} \text{ such that } p|n$$

   (a) Write out both statements in full sentences.

   (b) Interpret both statements, and explain the difference in meaning between the two. In particular, determine which of the statements is true and which is false.

   (c) Write the negations of both statements.

8. Write an explicit proof of the assertion made in Theorem 2: if an integer a is odd, then any positive power of a is odd. *Hint:* You can define an odd integer to be an integer of the form $(2b + 1)$ for some $b \in \mathbb{Z}$. You could complete this proof using the binomial theorem, or, alternatively, by mathematical induction.

9. (a) Prove by induction that for every $n \geq 1$,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

   (b) Prove by induction that for any integer $n > 1$, a set of n elements has exactly 2^n distinct subsets. (Note that we always consider the empty set $\varnothing$ to be a subset.)