Name: ______________________ Score (Out of 9 points):

A non-programmable, non-scientific calculator may be used.

1. (a) [2 points] Use the Euclidean algorithm to compute the greatest common divisor of 170 and 245.

We use the Euclidean algorithm:

$$\begin{aligned} 245 &= 1(170) + 75 \\ 170 &= 2(75) + 20 \\ 75 &= 3(20) + 15 \\ 20 &= 1(15) + 5 \\ 15 &= 3(5) \end{aligned}$$

The gcd is the last nonzero remainder; $\gcd(245, 170) = 5$.

(b) [2 points] Find integers u and v such that $245u + 170v = \gcd(245, 170)$.

Working backward through our Euclidean algorithm computation:

$$\begin{aligned} 5 &= 20 - 15 && \\ 5 &= 20 - (75 - 3(20)) && \text{[sub } 15 = 75 - 3(20)\text{]} \\ 5 &= 4(20) - (75) && \text{[collect (20) terms]} \\ 5 &= 4(170 - 2(75)) - (75) && \text{[sub } 20 = 170 - 2(75)\text{]} \\ 5 &= 4(170) - 9(75) && \text{[collect (75) terms]} \\ 5 &= 4(170) - 9(245 - 170) && \text{[sub } 75 = 245 - 170\text{]} \\ 5 &= 13(170) - 9(245) && \text{[collect (170) terms]} \end{aligned}$$

Thus we have found a solution $u = -9$ and $v = 13$.

 The quiz continues on the back of the page.

2. Suppose that $a, b \in \mathbb{Z}$ and that $\gcd(a, b) = d$. Let $c > 0$ be a common divisor of a and b. Recall that we proved that $\frac{d}{c}$ is an integer.

   (a) [4 points] Using the definitions of divisibility and gcd, prove that $\gcd\left(\frac{a}{c}, \frac{b}{c}\right) = \frac{d}{c}$.

   To prove that $\gcd\left(\frac{a}{c}, \frac{b}{c}\right) = \frac{d}{c}$, we must show two things: that $\frac{d}{c}$ is a divisor of $\left(\frac{a}{c}\right)$ and $\left(\frac{b}{c}\right)$, and that it is the largest such common divisor.

   Claim: $\left(\frac{d}{c}\right)$ divides both $\left(\frac{a}{c}\right)$ and $\left(\frac{b}{c}\right)$.

   Proof. That $\gcd(a, b) = d$ means that d is a divisor of both a and b: by definition, there are integers n and m so that $a = md$ and $b = nd$. But then dividing by c,

   $$\left(\frac{a}{c}\right) = m\left(\frac{d}{c}\right) \qquad \text{and} \qquad \left(\frac{b}{c}\right) = n\left(\frac{d}{c}\right),$$

   so we conclude that $\left(\frac{d}{c}\right)$ is a divisor of both $\left(\frac{a}{c}\right)$ and $\left(\frac{b}{c}\right)$.

   □

   Claim: $\left(\frac{d}{c}\right)$ is the largest common divisor of $\left(\frac{a}{c}\right)$ and $\left(\frac{b}{c}\right)$.

   Proof. First note that $d =\gcd(a, b)$ must be positive, by virtue of being the *greatest* common divisor, so since $c > 0$ the integer $\left(\frac{d}{c}\right)$ must be positive.
   By a result from class, there exist integers u and v so that

   $$au + bv = \gcd(a, b) = d.$$

   Dividing through by c:

   $$\left(\frac{a}{c}\right)u + \left(\frac{b}{c}\right)v = \left(\frac{d}{c}\right)$$

   Any common divisor of $\left(\frac{a}{c}\right)$ and $\left(\frac{b}{c}\right)$ must divide the left-hand side of this equation, and hence divides $\left(\frac{d}{c}\right)$. Thus $\left(\frac{d}{c}\right)$ is at least as large as any divisor of $\left(\frac{a}{c}\right)$ and $\left(\frac{b}{c}\right)$. □

   This concludes the proof that $\gcd\left(\frac{a}{c}, \frac{b}{c}\right) = \frac{d}{c}$.

   (b) [1 point] Conclude as a special case that $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

   Taking the particular divisor $c = d$ of a and b, the above result gives $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = \frac{d}{d} = 1$