1. Let R be a ring, M and R–module and N an R–submodule of N.
   (a) Define the quotient module M/N, and prove that (in contrast to groups modulo subgroups) it always forms a well-defined R–module.
   (b) Show that $M\mathbb{N}$ satisfies the following universal property: If $\varphi : M \to Q$ is any map of R–modules satisfying $\phi(n) = 0$ for any $n \in N$, then φ factors uniquely through M/N.
   (c) Show that this universal property defines the quotient M/N uniquely up to unique isomorphism.

2. Suppose the following diagram is commutative and has exact rows. Prove that if m and p are injective, and l is surjective, then n is injective.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} \\ A' & \xrightarrow[r]{} & B' & \xrightarrow[s]{} & C' & \xrightarrow[t]{} & D' \end{array}$$

3. Let k be a field, and x, y indeterminates. Prove or disprove the following isomorphism of k–modules: $k[x, y] \cong k[x] \otimes_k k[y]$.

4. Let R be commutative and let M, N be R–modules. Show that there is a canonical isomorphism

$$M \otimes_R N \cong N \otimes_R M.$$

5. Let M, M_i be right R–modules and N, N_i be left R–modules. Use the universal property of the tensor product and the universal property of the direct sum to prove the following isomorphisms of abelian groups:

$$(M_1 \oplus M_2) \otimes_R N \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N) \qquad M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2)$$

6. Let V and W be vector spaces over a field $\mathbb{F}$ with bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$, respectively.
   (a) Show that $\{e_i \otimes f_j\}_{i=1,j=1}^{n,m}$ is a basis for $V \otimes_{\mathbb{F}} W$.
   (b) It follows from part (a) that any element α of $V \otimes_{\mathbb{F}} W$ can be written in the form $\alpha = \sum_{i,j} c_{i,j}(e_i \otimes f_j)$. Prove that α can be expressed as a simple tensor (that is, in the form $v \otimes w$ for $v \in V, w \in W$) if and only if the matrix $(c_{i,j})$ has rank 1.

7. Classify (up to conjugacy) all linear maps $T : \mathbb{Q}^5 \to \mathbb{Q}^5$ with characteristic polynomial $c(x) = x^2(x-2)^3$.

8. Let R be an integral domain, and consider a short exact sequence of finite-rank R–modules:

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$

   Show that rank(B) = rank(A)+ rank(C).

9. Let M be a finitely generated module over a PID R. Give necessary and sufficient conditions on the elementary divisors of M for M to *irreducible*.

10. Let M be a simple R–module. Prove that M is cyclic.

11. Let V be a finite dimensional complex vector space and $T : V \to V$ a linear map. Under what conditions is the associated $\mathbb{C}[x]$–module V completely irreducible?

12. Prove that 3×3 matrices over a field k are similar if and only if they have the same minimal and characteristic polynomials.

13. Determine representatives for all the conjugacy classes of $GL_2(\mathbb{F}_3)$.

14. Prove that any matrix A is similar to its transpose A^T.

15. Determine the rational and Jordan canonical form of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use these results to compute its characteristic and minimal polynomials, invariant factors, elementary divisors, eigenvalues, and dimensions of its (generalized) eigenspaces.

16. Prove that a linear map is diagonalizable if and only if its minimal polynomial has distinct roots.

17. Let k be a field and V a vector space over k. Prove that any group representation $G \to GL(V)$ extends uniquely to a map of rings $k[G] \to \mathrm{End}(V)$. Explain how this defines a $k[G]$–module structure on V.

18. Prove that there is a bijective correspondence between degree-1 representations of a group G, and degree-1 representations of its abelianization $G/[G,G]$.

19. Let G be a finite group, and $\mathbb{F}$ a field containing $\frac{1}{|G|}$.
    (a) State Maschke's theorem.
    (b) Show by example that if $|G|$ divides the characteristic of $\mathbb{F}$, then not all G–representations over $\mathbb{F}$ are completely reducible.

20. Prove that if G is a nontrivial group and $\mathbb{F}$ a field, then every irreducible $\mathbb{F}[G]$–module has dimension $< |G|$.

21. Prove that isomorphic G–representations have the same character.

22. Let V be a G-representation. Show that the action of a group element $g \in G$ on V is G–equivariant if and only if g is in the center of G.

23. Prove that if U is a complex irreducible representation of G, and $V = U \oplus U$, then there are infinitely many ways that V can be decomposed into two copies of U. What is $\mathrm{Hom}_{\mathbb{C}[G]}(U,V)$? $\mathrm{Hom}_{\mathbb{C}[G]}(V,U)$?

24. (a) Let $\mathbb{F}$ be a field. Given any finite set $B = \{b_1, \ldots, b_m\}$, with an action of G, show how to construct a permutation representation by G on the vector space over $\mathbb{F}$ with basis B. Show that each G-orbit of B corresponds to a G subrepresentation of V.
    (b) Suppose that G acts transitively on the basis B (more generally, you can reply this result to the span of each G-orbit of B). Show that the diagonal subspace $D = \langle b_1 + b_2 + \cdots + b_m \rangle$ is invariant under G, and that G acts on it trivially. Show the orthogonal complement of D,

$$D^\perp = \left\{ a_1 b_1 + \ldots + a_m b_m \;\middle|\; \sum a_i = 0 \right\}$$

    is also invariant under the action of G, so that V decomposes as a direct sum of G subrepresentations $V \cong D \oplus D^\perp$. Compute the degrees of D and $D^\perp$.
    (c) Suppose that G acts transitively on the basis B. Prove that $D^\perp$ does not contain any vectors fixed by G (and therefore does not contain any trivial subrepresentations).
    (d) Conclude that the regular representation $V \cong \mathbb{F}[G]$ decomposes into a direct sum of invariant subspaces:

$$\left\{ \sum_{g \in G} a e_g \;\middle|\; a \in \mathbb{F} \right\} \bigoplus \left\{ \sum_{g \in G} a_g e_g \;\middle|\; \sum_{g \in G} a_g = 0 \right\}$$

(e) Use this decomposition and the averaging map to give a new proof that the multiplicity of the trivial representation in $\mathbb{F}[G]$ is 1.

25. Let V be a vector space over $\mathbb{F}$ with basis $x_1, \ldots, x_n$. Construct an isomorphism of rings $\mathrm{Sym}^* V \cong \mathbb{F}[x_1, x_2, \ldots, x_n]$ that commutes with scalar multiplication by $\mathbb{F}$.

26. Let M be a module over a commutative ring R. Show that the constructions T^*M, $\mathrm{Sym}^* M$, and $\wedge^* M$ define functors from R–modules to rings (in fact, R–algebras).

27. Let V be a vector space over a field $\mathbb{F}$ of characteristic zero.

 (a) Show that you can identify $\mathrm{Sym}^* M$, and $\wedge^* M$ as **subspaces** of T^*M via the maps

$$x_1 x_2 \cdots x_k \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \qquad \text{and} \qquad x_1 \wedge x_2 \wedge \cdots \wedge x_k \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_k)$$

 (b) Show that $V \otimes_{\mathbb{F}} V \cong \mathrm{Sym}^2(V) \oplus \wedge^2 V$.

 (c) Show that $V \otimes_{\mathbb{F}} V \otimes_{\mathbb{F}} V \supsetneq \mathrm{Sym}^3(V) \oplus \wedge^3 V$.

28. Prove that a finite group G is abelian if and only if all its complex irreducible representations are 1-dimensional.

29. Let G be a finite group. Prove that every short exact sequence of finite dimensional $\mathbb{C}[G]$–modules splits. What conditions on a field $\mathbb{F}$ would ensure that every short exact sequence of finite $\mathbb{F}$–dimensional $\mathbb{F}[G]$–modules split?

30. Let V be an irreducible complex representation of a finite group G. Show that the multiplicity of V in a G–representation U is equal to $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(V, U) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(U, V)$.

31. It is a nonobvious fact that all values of the irreducible complex characters of the symmetric groups are integers. Prove that if V is an irreducible representation of S_n of degree at least 2, then there must be at least one conjugacy class of S_n where χ_V takes on the value zero.

32. (a) Let g be a diagonalizable linear transformation acting on a vector space V, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Describe the set of eigenvalues of the map induced by g on the spaces $V \otimes V$, $V \otimes V \otimes V$, $\wedge^2 V$, $\wedge^3 V$, $\mathrm{Sym}^2 V$, and $\mathrm{Sym}^3 V$.

 (b) Suppose G is a finite group, and V a representation of G. Derive formulas for the characters of the representations $V \otimes V$, $V \otimes V \otimes V$, $\wedge^2 V$, $\wedge^3 V$, $\mathrm{Sym}^2 V$, and $\mathrm{Sym}^3 V$ in terms of the character χ_V for V.

33. Consider the complex S_4–representation $\mathbb{C}^4 \cong \underline{\mathrm{Trv}} \oplus \underline{\mathrm{Std}}$.

 (a) Prove that $\underline{\mathrm{Std}}$ is irreducible.

 (b) Compute the character table of S_4.

 (c) Compute the characters of $\wedge^3 \mathbb{C}^3$ and $\mathrm{Sym}^2 \underline{\mathrm{Std}}$, and compute their decompositions into irreducible representations.

34. Let G be a group. When are two 1-dimensional representations of G isomorphic?

35. Let G be a finite group.

 (a) Prove that the dimension of the space of class functions $G \to \mathbb{F}$ over $\mathbb{F}$ is equal to the number of conjugacy classes of G.

 (b) Prove that a complex-valued class function on G is a character if and only if it is a nonnegative integer linear combination of irreducible characters.

36. Let G be a finite group and C be its character table (of all irreducible characters).

(a) Show that the "orthogonality of characters" result is equivalent to the statement that the matrix C satisfies the relation $\overline{C}DC^T = Id$ for a certain diagonal matrix D. What is D?

(b) Conclude from this equation that $C^T\overline{C} = D^{-1}$. Use this equation to derive the second orthogonality result for characters.

(c) Explicitly verify the relations $\overline{C}DC^T = Id$ and $C^T\overline{C} = D^{-1}$ for the character table for S_3.

37. (a) State the two versions of the orthogonality results for the complex character table of a finite group G (the version for rows, and the version for columns).

    (b) Describe the $\mathbb{C}$–vector space of class functions on G, and explain why this space is an inner product space. Explain how this inner product structure relates to the our orthogonality theorem of characters.

    (c) Explain the utility of the orthogonality relations for decomposing G–representations into their irreducible components.

    (d) Write down the character table for S_3. Verify that all rows and columns satisfy the orthogonality relations.

38. Prove that the character table is an invertible matrix.

39. Let A be a finite abelian group.

    (a) Explain why the complex representations of A are precisely the set of group homomorphisms from A to the multiplicative group of units $\mathbb{C}^\times$ of $\mathbb{C}$.

    (b) Let $a \in A$ be an order of element k. What are the possible homomorphic images of a in $\mathbb{C}^\times$?

    (c) Let A be a finite cyclic group of order n. State the number of non-isomorphic representations of A, and describe these explicitly.

    (d) Let ξ_n denote an n^{th} root of unity. Write down the character tables for the groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

40. Let G be a group, and V and U be irreducible complex representations of G.

    (a) Show by example that $U \otimes_{\mathbb{C}} V$ may or may not be an irreducible G–representation.

    (b) Prove that if U is 1-dimensional, then $U \otimes_{\mathbb{C}} V$ is an irreducible G–representation.