Reading: Dummit–Foote Ch 10.2–10.3.

Recall: We assume that all rings have a multiplicative identity 1, that a subring of R must contain 1_R, and that a ring homomorphism $R \to S$ must map 1_R to 1_S.

Summary of definitions and main results

Definitions we've covered: Homomorphism of R–modules, isomorphism of R–modules, kernel, image, $\mathrm{Hom}_R(M, N)$, $\mathrm{End}_R(M)$, quotient of R–modules, sum of R–submodules.

Main results: R–linearity criterion for maps, kernels and images are R–submodules, for R commutative $\mathrm{Hom}_R(M, N)$ is an R–module, $\mathrm{End}_R(M)$ is a ring, factor theorem, four isomorphism theorems.

Warm-Up Questions

The "warm-up" questions do not need to be submitted (and won't be graded), however, you are responsible for understanding their solutions.

1. Define an R–module M by a ring homomorphism $R \to \mathrm{End}(M)$. We have another name for the kernel of this ring map – what is it?
2. We saw that a R–module structure on M can also be defined by a homomorphism of rings $R \to \mathrm{End}(M)$. From this perspective, give an alternate equivalent definition of the R–linear endomorphisms $\mathrm{End}_R(M)$.
3. (a) Prove the R–linearity criterion: $\phi : M \to N$ is an R–module map if and only if

$$\phi(rm + n) = r\phi(m) + \phi(n) \qquad \text{for all } m, n \in M \text{ and } r \in R.$$

   (b) Prove that the composition of R–module homomorphisms is again an R–module homomorphism.
   (c) Let $\phi : M \to N$ be an R–module homomorphism. Show that $\ker(\phi)$ is an R–submodule of M, and that $\mathrm{im}(\phi)$ is an R–submodule of N.
   (d) Show that an isomorphism of R–modules $\phi : M \to N$ has an inverse ϕ^{-1} which is also R–linear, and an isomorphism of R–modules $N \to M$.
   (e) Show that a homomorphism of R–modules ϕ is injective if and only if $\ker(\phi) = \{0\}$.
4. (a) Let M and N be R–modules. Show that every R–module map $M \to N$ is also a group homomorphism of the underlying abelian groups M and N.
   (b) Show that if R is a field, then R–module maps are precisely linear transformations of vector spaces.
   (c) Show that if $R = \mathbb{Z}$, then R–module maps are precisely group homomorphisms.
   (d) Show by example that a homomorphism of the underlying abelian groups M and N need not be a homomorphism of R–modules.
   (e) Now let $M = N$. Show that the set $\mathrm{End}(M) = \mathrm{End}_{\mathbb{Z}}(M)$ (the group endormophisms of the underlying abelian group M) and the set $\mathrm{End}_R(M)$ (the R–linear endomorphisms of the R–module M) may not be equal.
5. Consider R as a module over itself.
   (a) Show by example that not every map of R–modules $R \to R$ is a ring homomorphism.
   (b) Show by example that not every ring homomorphism is an R–module homomorphism.
   (c) Suppose that ϕ is both a ring map and a map of R–modules. What must ϕ be?
6. (a) For R–modules M and N, prove that $\mathrm{Hom}_R(M, N)$ is an abelian group, and $\mathrm{End}_R(M)$ is a ring.

(b) For a commutative ring R, what is the ring $\mathrm{End}_R(R)$?

(c) When R is commutative, show that $\mathrm{Hom}_R(M, N)$ is an R–module. What if R is not commutative?

(d) Let M be a right R–module. Prove that $\mathrm{Hom}_{\mathbb{Z}}(M, R)$ is a left R–module. What if M is a left R–module?

7. (a) Let M be an R–module. For which ring elements $r \in R$ will the map $m \mapsto rm$ define an R–module homomorphism on M?
   (b) Show that if R is commutative then there is a natural map of rings $R \to \mathrm{End}_R(M)$.
   (c) Show by example that the map $R \to \mathrm{End}_R(M)$ may or may not be injective.
8. (a) Compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/12\mathbb{Z}, \mathbb{Z}/15\mathbb{Z})$ as a $\mathbb{Z}$–module.
   (b) For integers m, n, compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ as a $\mathbb{Z}$–module.
9. Let A and B be R–submodules of an R–module M.
   (a) Prove that the sum $A + B$ is an R–submodule of M.
   (b) Prove that $A+B$ is the smallest submodule of M containing A and B in the following sense: if any submodule N of M contains both A and B, then N contains $A + B$.
10. State and prove the four isomorphism theorems for modules (Section 10.2 Theorem 4.)
11. Use the first isomorphism theorem to prove that if $x \in R$ then the cyclic module Rx is isomorphic to the R–module $R/\mathrm{ann}(x)$. Deduce that if R is an integral domain, then $Rx \cong R$ as R–modules.
12. Show that the rank-nullity theorem for linear transformations of vector spaces is a consequence of the first isomorphism theorem for modules.
13. Let R be a ring. A left ideal I in R is *maximal* if the only left ideals in R containing I are I and R. Use the fourth isomorphism theorem to show that R/I is *simple* (it has no proper nontrivial submodules).

Assignment Questions

The following questions should be handed in.

1. Let $\phi : M \to N$ be a homomorphism of R–modules. Let I be a right ideal of R. Let $\mathrm{ann}_M(I)$ denote the annihilator of I in M, and $\mathrm{ann}_N(I)$ the annihilator of I in N. Prove or find a counterexample: $\phi(\mathrm{ann}_M(I)) \subseteq \mathrm{ann}_N(I)$.
2. Let R be a commutative ring and N an R–module.
   (a) Prove that there is an isomorphism of left R–modules $N \cong \mathrm{Hom}_R(R, N)$.
   (b) Let n be a positive integer. Compute $\mathrm{Hom}_R(R^n, N)$.
   (c) What can you say about $\mathrm{Hom}_R(N, R)$?
3. For any ring R and positive integer n, show that $\mathrm{End}_R(R^n)$ is isomorphic (as a ring) to the ring of $n \times n$ matrices with entries in R.
4. For R–modules M, N, P, there is a composition map $\mathrm{Hom}_R(M, N) \times \mathrm{Hom}_R(N, P) \longrightarrow \mathrm{Hom}_R(M, P)$ given by $(f, g) \longmapsto g \circ f$.
   (a) When R is commutative, is this map a homomorphism of R–modules?
   (b) Give examples of a ring R and distinct R–modules M, N, P such that this map is surjective, and where this map is not surjective.
5. Let $R[x, y]$ be the polynomials in indeterminates x and y over a commutative ring R. Use the isomorphism theorems to prove the following isomorphisms of R–modules.

(a) $R[x,y]/\langle x\rangle \cong R[y]$.

(b) Let $p(x,y)$ be a polynomial in x and y. Then $R[x,y]/\langle x, p(x,y)\rangle \cong R[y]/\langle p(0,y)\rangle$.

(c) Let $q(x)$ be a polynomial in x. Then $R[x,y]/\langle y - q(x)\rangle \cong R[x]$.

6. Let U, V, W be vector spaces over a field $\mathbb{F}$. Let $\phi : U \to V$ be an injective linear map, and let $\psi : V \to W$ be a surjective linear map. Prove that both ϕ and ψ have one-sided inverses. Show by example that when R is not a field, not all surjective maps of R–modules have (one-sided) inverses, and show that not all injective maps of R–modules have (one-sided) inverses.