Reading: Dummit–Foote Ch 10.3 & pp 911-912.

Summary of definitions and main results

Definitions we've covered: generators of an R–module, the R–submodule RA generated by a set A, finite generation, cyclic module, minimal set of generators, Noetherian ring, direct product, direct sum (externel and internal), free module, basis, rank of a free module, category, object, morphism, monomorphism, epimorphism, isomorphism, universal property

Main results: Equivalent definitions of (internal) direct sums, construction of the free module $F(A)$, universal property of free modules, universal properties define objects up to unique isomorphism, in the category R-mod monomorphisms are precisely the injections.

Warm-Up Questions

The "warm-up" questions do not need to be submitted (and won't be graded), however, you are responsible for understanding their solutions.

1. Find an example of an R-module M that is isomorphic as R–modules to one of its proper submodules.
2. Let A and B be submodules of the R–module M. Show that $A+B$ is equal to $R(A \cup B)$, the submodule generated by $A \cup B$, as R–submodules of M.
3. Let R be a ring and I a two-sided ideal of R. For each of the following R–modules M indicate whether M is finitely generated, cyclically generated, or more information is needed:
   $M = R^n$ for $n \in \mathbb{N}$, polynomials $M = R[x]$, series $M = R[[x]]$, $M = I$, and $M = R/I$.
4. (a) Prove that if M is a finitely generated R–module, and $\phi : M \to N$ a map of R–modules, then its image $\phi(M)$ is finitely generated by the images of the generators. Conclude in particular that all quotients of finitely generated modules are finitely generated.
   (b) Let M be an R–module and N a submodule. Prove that if both N and M/N are finitely generated R–modules, then M is a finitely generated R–module.
5. (a) Let A be any finite set of n elements. Show that the free R–module on A is isomorphic as an R–module to R^n.
   (b) For R commutative, are the polynomial rings $R[x]$ and $R[x, y]$ free R–modules? What about Laurent polynomials $R[x, x^{-1}]$? Rational functions in x?
   (c) Do these arguments work for series $R[[x]]$?
6. In class (and in Dummit-Foote 10.3 Theorem 6) we gave a construction of a free module $F(A)$ on a set A. Verify that this construction is in fact a free module with basis A (as given in the definition on p354).
7. (a) Let M be an R–module generated by a set $A \subseteq M$. Show that there is a unique R–module map $F(A) \to M$ that restricts to the identity map on the set A, and that this map is surjective.
   (b) Conclude that an R–module M is finitely generated if and only if it admits a surjection from a finitely generated free module.
8. (a) Citing results from linear algebra, explain why every vector space over a field $\mathbb{F}$ is a free $\mathbb{F}$–module.
   (b) When $\mathbb{F}$ is a field, any minimal finite generating set $B = \{a_1, \ldots a_n\}$ of an $\mathbb{F}$–module must be linearly independent and therefore a basis. Prove that in general, if an R–module has a minimal generating set $B = \{a_1, \ldots a_n\}$, then R need not be free on B.

(c) Suppose that M is an R–module containing elements $\{a_1, a_2, \ldots, a_n\}$ such that $M = Ra_1 \oplus Ra_2 \oplus \cdots \oplus Ra_n$. Explain how $A = \{a_1, a_2, \ldots a_n\}$ could fail to be a basis for M. What conditions on the elements a_i could ensure that A is a basis?

9. (a) Citing results from linear algebra, explain why every field $\mathbb{F}$ is Noetherian.

   (b) Citing results from group theory, explain why $\mathbb{Z}$ is Noetherian.

10. Show that $M = \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ is a free $\mathbb{Z}/10\mathbb{Z}$–module by finding a basis. Show that the element $(2, 2)$ cannot be an element of any basis for M. Is the submodule $N = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ also free?

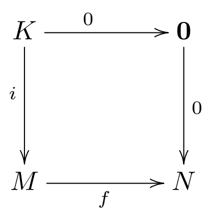
11. Let $\{M_i \mid i \in I\}$ be a (possibly infinite) set of R–modules. Prove that the direct sum $\bigoplus_{i \in I} M_i$ is a submodule of the direct product $\prod_{i \in I} M_i$, but show by example that these may not be isomorphic in general. *Hint*: What are their cardinalities?

12. (a) Prove that in the category of R–modules, a morphism is epic if and only if it is a surjective map.

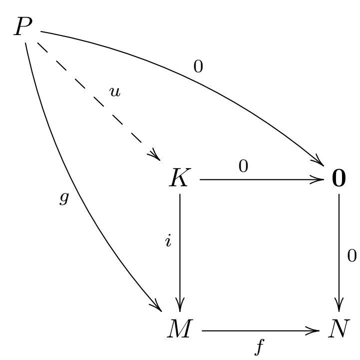
    (b) Prove that in the category of rings, the map $\mathbb{Z} \to \mathbb{Q}$ is an epic morphism that is not surjective.

13. (a) A *zero object* $\mathbf{0}$ in a category is an object with the following property: For any object M, there is a unique morphism from M to $\mathbf{0}$, and a unique morphism from $\mathbf{0}$ to M. Let $\mathcal{C}$ be the category of R–modules, and show that the zero object is the zero module $\{0\}$. This definition allows us to define the *zero map* 0 between R–modules M and N: it is the composition of the unique map $M \to \mathbf{0}$ with the unique map $\mathbf{0} \to N$.

    (b) Let $\mathcal{C}$ be the category R–modules. Verify that the kernel of an R–module map satisfies the following universal property. If $f : M \to N$ is a morphism in $\mathcal{C}$, then define the *kernel* $i : K \to M$ of f to be the map i such that $f \circ i$ is the zero morphism 0



    and satisfying the following: whenever there is a map of R–modules $g : P \to M$ such that $f \circ g = 0$, there is a unique map $u : P \to K$ such that $i \circ u = g$. In other words, there is a unique map u that makes the following diagram commute.



    (c) Explain why this univeral property determines the map $i : K \to M$ up to unique isomorphism. Conclude that this universal property can be taken as the definition of the kernel of an R–module map.

Assignment Questions

1. Let V be an $\mathbb{C}[x]$–module with V finite dimensional over $\mathbb{C}$, and x acting by the linear map T. For which linear maps T will V be cyclically generated? Give conditions on the eigenvalues and eigenspaces of T.

2. Suppose a finitely generated R–module M has a minimal generating set $A = \{a_1, a_2, \ldots, a_n\}$. Prove or find a counterexample: $M \cong Ra_1 \oplus Ra_2 \oplus \cdots \oplus Ra_n$.

3. Let R be a ring. Show that an arbitrary direct sum of free R–modules is free, but an arbitrary direct product need not be. *Hint:* Dummit–Foote 10.3 # 24.

4. (a) Let $M_1, \ldots M_n$ be R–modules, and N_i a submodule of M_i for all i. Prove that

$$\frac{M_1 \times M_2 \times \cdots \times M_n}{N_1 \times N_2 \times \ldots \times N_n} \cong \left(\frac{M_1}{N_1}\right) \times \left(\frac{M_2}{N_2}\right) \times \cdots \times \left(\frac{M_n}{N_n}\right).$$

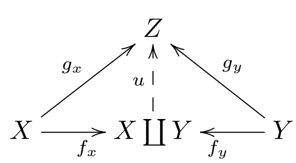
   (b) Let I be any left ideal of R, and let $IR^n = \{\text{finite sums } \sum a_i x_i \mid a_i \in I, x_i \in R^n\}$. Prove that

$$\frac{R^n}{IR^n} \cong \frac{R}{IR} \times \frac{R}{IR} \times \cdots \times \frac{R}{IR}.$$

   (c) Let R be a commutative ring, and let $n, m \in \mathbb{N}$. Prove that that $R^n \cong R^m$ if and only if $n = m$. You can assume without proof that finite-dimensional vector spaces are isomorphic if and only if their dimensions are equal. You may also assume Zorn's Lemma.

   (d) Show that when R is not commutative, this statement is false – that is, free R–modules need not have a unique rank. *Hint:* See Dummit–Foote 10.3 # 27.

5. Let $\mathcal{C}$ be a category with objects X and Y. The *coproduct* of X and Y (if it exists) is an object $X \coprod Y$ in $\mathcal{C}$ with maps $f_x : X \to X \coprod Y$ and $f_y : Y \to X \coprod Y$ satisfying the following universal property: whenever there is an object Z with maps $g_x : X \to Z$ and $g_y : Y \to Z$, there exists a unique map $u : X \coprod Y \to Z$ that makes the following diagram commute:



   (a) Prove that in the category of R–modules, the coproduct of R–modules $X \coprod Y$ is $X \oplus Y$ with the canonical inclusions of X and Y. In other words, this universal property defines the direct sum operation on R–modules.

   (b) Prove that in the category of groups, the univeral property for the coproduct $X \coprod Y$ of groups X and Y does *not* define the direct product of those groups. (It is a construction called the *free product* of groups).

   (c) Prove that in the category of sets, the coproduct $X \coprod Y$ of sets X and Y is their disjoint union.