Reading: Dummit–Foote pp 811-813, Ch 10.5 up to (and including) Theorem 38 on p398 (the book goes into more detail on these topics than we will).

Summary of definitions and main results

Definitions we've covered: covariant and contravariant functors, forgetful functor, free functor, exact, exact sequence, short exact sequence, extension of C by A, extension problem, presentation, relations, homomorphism of short exact sequences, isomorphism of short exact sequences, equivalence of extensions, split short exact sequence, exact functor, right exact functor, left exact functor.

Main results: Free functor $F : \underline{\text{Set}} \to R\text{–}\underline{\text{Mod}}$ is functorial, Baby Five Lemma, Splitting Lemma, $\text{Hom}_R(D, -)$ is a covariant functor, $\text{Hom}_R(-, D)$ is a contravariant functor, $\text{Hom}_R(D, -)$ and $\text{Hom}_R(-, D)$ are left exact.

Warm-Up Questions

The "warm-up" questions do not need to be submitted (and won't be graded), however, you are responsible for understanding their solutions.

1. Under what conditions on the natural numbers m and n will the $\mathbb{Z}$–module $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ be cyclically generated?

2. Let R be a ring, and let R–$\underline{\text{Mod}}$ be the category of R–modules. Let $\underline{\text{Ab}}$ be the category of abelian groups. Show that there is a covariant functor $\mathscr{F} : R\text{–}\underline{\text{Mod}} \to \underline{\text{Ab}}$ that maps an R–module M to its underlying abelian group $\mathscr{F}(M)$. (This is an example of a *forgetful functor*, since it forgets the extra data of the action of R on M).

3. Given a group G, define a category $\mathscr{G}$ with a single object $\bigstar$ and morphisms $\text{Hom}_{\mathscr{G}}(\bigstar, \bigstar) = \{g \mid g \in G\}$. The composition law is given by the group operation. Show that a function between groups $G \to H$ is a group homomorphism if and only if the corresponding map between categories $\mathscr{G} \to \mathscr{H}$ is a functor.

4. Let $\underline{\text{f Set}}$ denote the category of finite sets and all functions between sets. Let $\mathscr{P} : \underline{\text{f Set}} \to \underline{\text{f Set}}$ be the function that takes a finite set A to its *power set* $\mathscr{P}(A)$, the set of all subsets of A. If $f : A \to B$ is a function of finite sets, let $\mathscr{P}(f) : \mathscr{P}(A) \to \mathscr{P}(B)$ be the function that takes a subset $U \subseteq A$ to the subset $f(U) \subseteq B$.
   (a) Show that $\mathscr{P}$ is a covariant functor.
   (b) What if we had instead defined $\mathscr{P}(f) : \mathscr{P}(B) \to \mathscr{P}(A)$ to take a subset $U \subseteq B$ to its preimage $f^{-1}(U) \subseteq A$ under f?

5. Let $\underline{\text{Grp}}$ be the category of groups and group homomorphisms. Let Z be the map $Z : \underline{\text{Grp}} \to \underline{\text{Grp}}$ that maps a group G to its centre $Z(G) = \{a \in G \mid ag = ga \ \ \forall g \in G\}$. Show that Z **cannot** be made into a functor by defining it to take a map of groups $f : G \to H$ to the restriction $f|_{Z(G)}$ of f to $Z(G)$, since $f(Z(G))$ may not be contained in $Z(H)$.

6. Let R be a ring. Consider the map on the objects of R–$\underline{\text{Mod}}$ that takes and R–module M to the submodule $\text{ann}(R)$, and takes a morphism of R–modules $f : M \to N$ to its restriction $f|_{ann(R)}$ to the submodule $\text{ann}(R) \subseteq M$. Does this give a well-defined functor $R\text{–}\underline{\text{Mod}} \to R\text{–}\underline{\text{Mod}}$?

7. Write down short exact sequences giving presentations of the following R–modules M. Give a list of generators and relations for M.
   (a) R^n
   (b) $R = \mathbb{Z}$, $M = \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$
   (c) $R = \mathbb{Q}$, $M = \mathbb{Q}[x]/\langle x^2 + 1 \rangle$

(d) $R = \mathbb{C}[x,y]$, $M = \langle x, y\rangle$

8. (a) Find two nonequivalent extensions of $\mathbb{Z}$–modules $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$.
   (b) Find two nonequivalent extensions of $\mathbb{Z}$–modules $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$.
   (c) How many extensions of $\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$ can you find?

9. (a) Show that if $0 \to U \to W \to V \to 0$ is a short exact sequence of vector spaces, then $W \cong V \oplus U$.
   (b) Show that any two extensions of vector spaces V by U are isomorphic.

10. Use the Splitting Lemma to show that if m and n are coprime, the following short exact sequence splits:

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/mn\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

What if m and n are not coprime?

11. Show by example that isomorphic extensions need not be equivalent. *Hint:* Page 382, Example (5).

Assignment Questions

1. Let N and M_i be R–modules for i in an index set I. Prove the following isomorphisms of abelian groups:
   (a) $\operatorname{Hom}_R\left(N, \prod_{i\in I} M_i\right) \cong \prod_{i\in I} \operatorname{Hom}_R(N, M_i)$
   (b) $\operatorname{Hom}_R\left(\bigoplus_{i\in I} M_i,\ N\right) \cong \prod_{i\in I} \operatorname{Hom}_R(M_i, N)$

2. **(Baby Five Lemma).** Consider a homomorphism of short exact sequences of R–modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\phi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow[\psi']{} & B' & \xrightarrow[\phi']{} & C' & \longrightarrow & 0 \end{array}$$

Prove the remaining step in the Baby Five Lemma: If α and γ both surject, then β must also surject.

3. **(The Splitting Lemma).** Let R be a ring, and consider the short exact sequence of R–modules:

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0.$$

Prove that the following are equivalent.

(i) The sequence *splits*, that is, B is isomorphic to $A \oplus C$ such that ψ corresponds to the natural inclusion of A, and ϕ corresponds to the natural projection onto C.

(ii) There is a map $\phi' : C \to B$ such that $\phi \circ \phi'$ is the identity on C.

$$0 \longrightarrow A \xrightarrow{\psi} B \underset{\phi'}{\overset{\phi}{\rightleftarrows}} C \to 0$$

(iii) There is a map $\psi' : B \to A$ such that $\psi' \circ \psi$ is the identity on A.

$$0 \to A \underset{\psi'}{\overset{\psi}{\rightleftarrows}} B \xrightarrow{\phi} C \longrightarrow 0$$

The maps ϕ' and ψ' are called *splitting homomorphisms*.

4. We proved in class that the map $\mathrm{Hom}_R(D, -) : R\text{–}\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}}$ is a covariant, left exact functor.
   (a) What is the functor of $\mathbb{Z}$–modules $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$?
   (b) To which finitely-generated abelian groups does the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ map the $\mathbb{Z}$–modules $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $(\mathbb{Z}/n\mathbb{Z})^p$, $\mathbb{Z}/n^p\mathbb{Z}$, and $\mathbb{Z}/m\mathbb{Z}$ (for m, n coprime)?
   (c) Describe the sequence of abelian groups and the maps obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$ to the following short exact sequences:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

   (d) **(Projective R-modules)**. An R–module P is called *projective* if the functor $\mathrm{Hom}_R(P, -)$ is an exact functor. Prove that if P is a free R–module, then P is projective. In other words, show that if P is free and $\phi : M \to N$ is any surjective map of R–modules, then the induced map

$$\phi_* : \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$$

   is surjective.
   (e) Given any positive integer n, show that $\mathbb{Z}/n\mathbb{Z}$ is **not** a projective $\mathbb{Z}$–module.

5. **(The functor $\mathrm{Hom}_R(-, D)$)**.
   (a) Show that if D is any R–module, then there is a **contravariant** functor

$$\begin{aligned} \mathrm{Hom}_R(-, D) : R\text{–}\underline{\mathrm{Mod}} &\longrightarrow \underline{\mathrm{Ab}} \\ M &\longmapsto \mathrm{Hom}_R(M, D) \\ \{\phi : M \to N\} &\longmapsto \left\{ \begin{smallmatrix} \phi^* : \mathrm{Hom}_R(N,D) \longrightarrow \mathrm{Hom}_R(M,D) \\ f \longmapsto f \circ \phi \end{smallmatrix} \right\} \end{aligned}$$

   (b) Show that $\mathrm{Hom}_R(-, D)$ is left exact. This means (for a contravariant functor) that for any short exact sequence

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \to 0$$

   the following is exact:

$$0 \longrightarrow \mathrm{Hom}_R(C, D) \xrightarrow{\phi^*} \mathrm{Hom}_R(B, D) \xrightarrow{\psi^*} \mathrm{Hom}_R(A, D).$$

   (c) Give an example of an injective R–module map $\phi : M \to N$ such that ϕ^* is not injective.
   Remark: An R–module I is called *injective* if $\mathrm{Hom}_R(-, I)$ is exact.

Hint: Many results of Exercises 4 and 5 are addressed in Dummit-Foote 10.5.