Reading: Dummit–Foote Ch 10.4, Ch 10.5 p398-402.

Summary of definitions and main results

Definitions we've covered: tensor products, (S, R)–bimodule and R–bimodule, R-balanced map, R-bilinear map, universal property of the tensor product, extension of scalars, tensor product of R-linear maps.

Main results: Explicit construction of $M \otimes_R N$, verification that it satisfies the universal property, criteria for S-module structures on tensor products, $R/I \otimes_R N \cong N/IN$, tensor product distributes over direct sums, $R^n \otimes_R N \cong N^n$, tensor product is associative, $D \otimes_R -$ is a right-exact covariant functor, how to use the universal property (or right exactness) to compute tensor products in specific examples, hom-tensor adjunction

Warm-Up Questions

1. Let $\mathscr{C}$ be a category containing objects A and B, and let F be a functor $F : \mathscr{C} \to \mathscr{D}$. Show that if A and B are isomorphic objects of $\mathscr{C}$, then $F(A)$ and $F(B)$ will be isomorphic objects of $\mathscr{D}$.

2. Let 0 denote the trivial abelian group. Give an example of a functor $F : \underline{\text{Ab}} \to \underline{\text{Ab}}$ such that $F(0) = 0$, and a functor $F : \underline{\text{Ab}} \to \underline{\text{Ab}}$ such that $F(0) \neq 0$.

3. Explain why, when R is commutative, a left R–module M will also be a right R–module under the action $mr = rm$, and conversely any right R–module N will also have an induced left R–module structure. Will these actions automatically give an R–bimodule structure? Why will these constructions generally not work when R is non-commutative?

4. Let R be a ring with right R–module M and left R–module N. Show that the natural map
$$M \times N \longrightarrow M \otimes_R N$$
is **not** a group homomorphism. What are the constraints on this map, as imposed by the defining relations of $M \otimes_R N$?

5. Let R and S be rings (possibly the same ring). Let M be a right R–module and N a left R–module. When will the tensor product $M \otimes_R N$ have the structure of an abelian group, and under what conditions will it additionally have the structure of an S–module?

6. Let R be a ring with right R–module M and left R–module N. Which of the following maps are R–balanced? Which are homomorphisms of abelian groups? For the maps that are R–balanced, describe how they factor through the tensor product.
   (a) The identity map $M \times N \longrightarrow M \times N$.
   (b) The natural projections of $M \times N$ onto M and N.
   (c) The natural map $M \times N \longrightarrow M \otimes_R N$.
   (d) Suppose M and N are ideals of R. The multiplication map
   $$\begin{aligned} M \times N &\longrightarrow R \\ (m, n) &\longmapsto mn \end{aligned}$$
   (e) Suppose R is commutative. The matrix multiplication map
   $$\begin{aligned} M_{n\times k}(R) \times M_{k\times m}(R) &\longrightarrow M_{n\times m}(R) \\ (A, B) &\longmapsto AB \end{aligned}$$

(f) Suppose R is commutative and M, N, P are R–modules. The composition map:

$$\begin{aligned} \mathrm{Hom}_R(M,N) \times \mathrm{Hom}_R(N,P) &\longrightarrow \mathrm{Hom}_R(M,P) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

(g) Suppose R is commutative. The dot product map:

$$\begin{aligned} R^n \times R^n &\longrightarrow R \\ (v,w) &\longmapsto v \cdot w \end{aligned}$$

(h) Suppose R is commutative. The cross product map:

$$\begin{aligned} R^3 \times R^3 &\longrightarrow R^3 \\ (v,w) &\longmapsto v \times w \end{aligned}$$

(i) Suppose R is commutative. The determinant map:

$$\begin{aligned} R^2 \times R^2 &\longrightarrow R \\ (v,w) &\longmapsto \det \begin{bmatrix} | & | \\ v & w \\ | & | \end{bmatrix} \end{aligned}$$

7. Let R be a ring with right R–module M and left R–module N.
   (a) What is the additive identity in $M \otimes_R N$? Show that the simple tensors $0 \otimes n$ and $m \otimes 0$ will be zero in any tensor product $M \otimes_R N$.
   (b) Show there are always maps of abelian groups $N \to M \otimes_R N$, but that these maps may not be injective.

8. Let $V \cong \mathbb{C}^2$ be a complex vector space, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with respect to the standard basis e_1, e_2. Write down the matrix for the linear map induced by A on the four-dimensional vector space $V \otimes V$ with respect to the basis $e_1 \otimes e_1,\ e_1 \otimes e_2,\ e_2 \otimes e_1,\ e_2 \otimes e_2$.

9. Let V be a complex vector space. Let $T : V \to V$ be a diagonalizable linear map with eigenbasis $v_1, v_2, \ldots v_n$, and associated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. What are the eigenvalues of the map induced by T on $V \otimes V$, and what are the associated eigenvectors?

10. Let R be a ring and S a subring.
    (a) Give an example of R, S and an S–module that embeds into a R–module.
    (b) Give an example of R, S, and an S–module that cannot embed into any R–module.

11. (a) Suppose that A is a finite abelian group. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$.
    (b) Suppose that B is a finitely-generated abelian group. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} B$ is a $\mathbb{Q}$–vector space. What determines its dimension?

12. Let M be a right R–module and $N_1, \ldots, N_n$ a set of left R–modules. Verify that the tensor product distributes over direct sums (Dummit–Foote 10.4 Theorem 17). There is a unique group isomorphism

$$M \otimes_R (N_1 \oplus \cdots \oplus N_n) \cong (M \otimes_R N_1) \oplus \cdots \oplus (M \otimes_R N_n).$$

    Conclude that if N is a left R–module, $R^n \otimes_R N \cong N^n$.

13. (a) Let R be a ring, I a left ideal of R, and N a left R–module. Prove that $R/I \otimes_R N \cong N/IN$.
    (b) Let R be a commutative ring with ideals I and J. Prove the isomorphism of R–modules:

$$\begin{aligned} R/I \otimes_R R/J &\longrightarrow R/(I+J) \\ (r+I) \otimes (s+J) &\longmapsto rs + (I+J) \end{aligned}$$

14. Verify the associativity of the tensor product (Dummit-Foote 10.4 Theorem 14).

Assignment Questions

1. (a) For integers $m, n > 1$, compute the abelian groups $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$.
   (b) For integer $n > 1$, compute the abelian groups $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ and $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.
   (c) Compute the rational vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.
   (d) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are **not** isomorphic as vector spaces over $\mathbb{R}$.
   (e) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ **are** isomorphic as vector spaces over $\mathbb{Q}$.

   Note: By "compute" an abelian group I mean describe the group in terms of the classification of finitely generated abelian groups, as a product of cyclic groups. By "compute" a vector space I mean determine its dimension.

2. Let R be a ring, let A be a right R–module and B a left R–module. Prove that the universal property of the tensor product defines $A \otimes_R B$ uniquely up to unique isomorphism.

3. **(The functor $D \otimes_R -$ is right exact.)** Let R be any ring, and D a right R–module.
   (a) Show that the following map of categories is well-defined and a covariant functor:

$$\begin{aligned} D \otimes_R - : R\text{–}\underline{\mathrm{Mod}} &\longrightarrow \underline{\mathrm{Ab}} \\ M &\longmapsto D \otimes_R M \\ \{f : M \to N\} &\longmapsto \left\{ \begin{matrix} f_* : D \otimes_R M \mapsto D \otimes_R N \\ f_*(d \otimes m) = d \otimes f(m) \end{matrix} \right\} \end{aligned}$$

   (b) Show that the functor $D \otimes_R -$ is right exact.

   Hint: Dummit-Foote 10.5 Theorem 39.

4. For any ring R and right R–module D, the functor $D \otimes_R -$ is right exact. A similar argument shows that for any left R–module D the functor $- \otimes_R D$ is right exact.
   (a) Use the right-exactness of the functor $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} -$ and the short exact sequence of $\mathbb{Z}$–modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

   to (re)compute $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$.
   (b) More generally, let R be a ring and I a two-sided ideal. Use the right exactness of $- \otimes_R N$ and the short exact sequence of R–modules

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

   to (re)prove the result: $R/I \otimes_R N \cong N/IN$.
   (c) Let k be a field and let $R = k[x, y]$. Give simple descriptions of the following tensor products, and determine their dimensions over k.

$$\frac{R}{\langle x \rangle} \otimes_R \frac{R}{\langle x - y \rangle} \qquad \frac{R}{\langle x \rangle} \otimes_R \frac{R}{\langle x - 1 \rangle} \qquad \frac{R}{\langle y - 1 \rangle} \otimes_R \frac{R}{\langle x - y \rangle}$$

5. **(The hom-tensor adjunction: the commutative case.)** Let R be a commutative ring, and suppose that A, B, and C are R–modules. (Since R is commutative, they are all naturally R–bimodules). Prove the following isomorphism of R–modules:

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R\big(A, \mathrm{Hom}_R(B, C)\big).$$