Reading: Dummit–Foote Ch 12.1–12.3. We will not cover the portions of these chapters on computational algorithms.

Summary of definitions and main results

Definitions we've covered: Jordan canonical form, Jordan block $J_{\lambda,k}$.

Main results: Existence and uniqueness of Jordan canonical form; Jordan canonical form classifies linear maps up to conjugacy.

Warm-Up Questions

1. (a) Define the *characteristic polynomial* of a linear map $T : V \to V$.
   (b) Show that if $\dim(V) = n$, then the characteristic polynomial of T is a monic degree n polynomial.
   (c) Show that λ is a root of characteristic polynomial if and only if it is an eigenvalue of T.
   (d) Show that if $\dim(V) = 2$, then the characteristic polynomial of T is $x^2 - \text{Trace}(T)x + \text{Det}(T)$.
   (e) Citing properties of the determinant, prove that the characteristic polynomial of a matrix is a conjugacy class invariant.

2. Let $T : V \to V$ be a linear map. Show that T is the zero map if and only if $Tv = 0$ for all $v \in V$. Conclude in particular that T satisfies a polynomial $p(x)$ if and only if $p(T)v = 0$ for all $v \in V$.

3. Let A be a block diagonal matrix with blocks $A_1, \dots A_m$. Show that for any exponent $p \in \mathbb{Z}_{\geq 0}$, the matrix A^p is block diagonal with blocks $A_1^p, \dots, A_m^p$.

4. (a) Let $T : V \to V$ be a linear map on a finite dimensional $\mathbb{F}$–vector space V. Describe how to construct an $\mathbb{F}[x]$–module corresponding to the map T. Explain the relationship between T and this $\mathbb{F}[x]$–module, and explain what we can infer about T from the invariant factor decomposition of this $\mathbb{F}[x]$–module.
   (b) Conversely, suppose you have an $\mathbb{F}[x]$–module U, which is finite dimensional as a vector space over $\mathbb{F}$. Describe how the $\mathbb{F}[x]$–module structure on U is equivalent to the data of a vector space U with a linear map $T : U \to U$. (How is T defined?)

5. Show that if $\mathbb{F}$ is an algebraically closed field, then the prime elements of $\mathbb{F}[x]$ are polynomials of the form $(x - \lambda)$, $\lambda \in \mathbb{F}$.

6. (a) Consider the $\mathbb{F}[x]$–module $\mathbb{F}[x]/(x-\lambda)^k$ with basis $(x-\lambda)^{k-1}, (x-\lambda)^{k-2}, \dots, (x-\lambda), 1$. Carefully explain why, with respect to this basis, the map "multiplication by x" acts on this module by the Jordan matrix
$$J_{\lambda,k} = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & 1 \\ & & & & \lambda \end{bmatrix}.$$
   (b) Consider the elementary divisor form of the structure theorem for a finitely generated torsion $\mathbb{F}[x]$–module V:
$$V \cong \frac{\mathbb{F}[x]}{(x-\lambda_1)^{k_1}} \oplus \frac{\mathbb{F}[x]}{(x-\lambda_2)^{k_2}} \oplus \cdots \oplus \frac{\mathbb{F}[x]}{(x-\lambda_\ell)^{k_\ell}}$$

Conclude that, in a suitably chosen basis, the map "multiplication by x" will correspond to the block diagonal matrix:

$$\begin{bmatrix} J_{\lambda_1,k_1} & & & \\ & J_{\lambda_2,k_2} & & \\ & & \ddots & \\ & & & J_{\lambda_\ell,k_\ell} \end{bmatrix}$$

(c) Explain why the uniqueness of the elementary divisor decomposition of V implies that (up to re-ordering of the blocks) this will be the only matrix in Jordan canonical form that represents the map "multiplication by x" in any basis.

(d) Explain why two linear maps are conjugate if and only if they have the same Jordan canonical form.

7. Figure out how to use your favourite mathematics software program to output the rational canonical form and Jordan canonical form a matrix. (For example, try typing the following into Wolfram alpha:

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jordan canonical form {{5, 4, 2, 1}, {0, 1, -1, -1}, {-1, -1, 3, 0}, {1, 1, -1, 2}} ).
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(This exercise is optional for this course, but a good mathematical tool to have for the future).

8. For each of the following $\mathbb{C}[x]$–modules, list the invariant factors, the elementary divisors, and write the rational canonical form and Jordan canonical form of the linear map "multiplication by x". State the minimal and characteristic polynomials.

(a) $V \cong \dfrac{\mathbb{C}[x]}{(x-1)^2} \oplus \dfrac{\mathbb{C}[x]}{(x-1)(x-2)}$

(b) $V \cong \dfrac{\mathbb{C}[x]}{(x-1)(x-2)(x-3)}$

(c) $V \cong \dfrac{\mathbb{C}[x]}{(x-1)} \oplus \dfrac{\mathbb{C}[x]}{(x-1)^2} \oplus \dfrac{\mathbb{C}[x]}{(x-1)^2}$

9. Compute the Jordan canonical form of the following matrices:

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & -1 & 3 \end{bmatrix}$$

10. Determine all possible Jordan canonical forms for linear maps with characteristic polynomial

$$(x-1)^3(x-2)^2.$$

11. (a) Suppose a complex matrix A satisfies the equation $A^2 = -2A - 1$. What are the possibilities for its Jordan canonical form?

(b) Suppose a complex matrix A satisfies $A^3 = A$. Show that A is diagonalizable. Would this result hold if A had entries in a field of characteristic 2?

12. Prove that an $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

13. Suppose that a polynomial $f(x) \in \mathbb{C}[x]$ has no repeated roots. Show that all linear maps with characteristic polynomial $f(x)$ are similar.

14. Fix $\lambda \in \mathbb{F}$. Show that the number of conjugacy classes of matrices with characteristic polynomial $(x-\lambda)^n$ is equal to the number of partitions of n.

Assignment Questions

1. (a) Let H be a subgroup of a group G. Show by example that there may be elements in H which are not conjugate in H, but are conjugate in G. What is the relationship between the conjugacy classes in H and the conjugacy classes in G?

   (b) Let $\mathbb{E}$ be a field and $\mathbb{F}$ a subfield of $\mathbb{E}$. Let A and B be $n \times n$ matrices with coefficients in $\mathbb{F}$. Use the theory of rational canonical form to show that A and B are conjugate in $M_n(\mathbb{E})$ if and only if they are conjugate in $M_n(\mathbb{F})$.

   Remark. This implies:

   - If two matrices A and B are conjugate, then they are conjugate by a matrix with coefficients in the smallest field over which the entries of A and B are defined.
   - Matrices that are not conjugate in $M_n(\mathbb{F})$ cannot become conjugate when we extend scalars to a field extension.
   - Suppose $\mathbb{F}$ is a field that is not algebraically closed (like $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{F}_q$). Two linear maps over $\mathbb{F}$ are conjugate if and only if they have the same Jordan canonical form (over the algebraic closure of $\mathbb{F}$) – even if their Jordan canonical form is not defined over $\mathbb{F}$.

2. (a) Prove that a linear map on a finite dimensional $\mathbb{F}$–vector space is diagonalizable over $\mathbb{F}$ if and only if its minimal polynomial has distinct roots, all contained in $\mathbb{F}$.

   (b) Let $P : V \to V$ be a projection matrix. (This means that there is some decomposition $V = W \oplus U$ such that $P(w + u) = w$ for all $u \in U, w \in W$). Show that $P^2 v = Pv$ for all $v \in V$, and use this relation to show that P is diagonalizable. What are its eigenvalues?

   (c) Let A be an $n \times n$ invertible complex matrix. Show that if A has finite order p then A is diagonalizable, and describe the possible eigenvalues of A.

   (d) Suppose that V is a finite dimensional vector space over $\mathbb{F}$, and $T : V \to V$ is a diagonalizable linear map. Show that the restriction of T to any T–invariant subspace $W \subseteq V$ will also be diagonalizable, and therefore W must be a direct sum of eigenspaces of T.

3. Let $T : V \to V$ be a linear map on a n-dimensional $\mathbb{F}$–vector space V. Let $e_1, \ldots, e_n$ be a basis for V corresponding to the Jordan canonical form of T. Let I denote the identity matrix.

   Recall that an *eigenvector* v of T with *eigenvalue* λ is defined to be a nonzero element of $\ker(\lambda I - T)$, and that the *eigenspace* E_λ is defined to be the subspace of V

   $$E_\lambda = \ker(\lambda I - T) = \{\text{eigenvectors of } T \text{ with eigenvalue } \lambda\} \cup \{0\}$$

   For an eigenvalue λ of T, define the *algebraic multiplicity* of λ to be the multiplicity of the root $(x - \lambda)$ in the characteristic polynomial of T, and the *geometric multiplicity* to be the $\dim_{\mathbb{F}}(E_\lambda)$.

   (a) Let $J_{\lambda,k}$ denote the $k \times k$ Jordan block with diagonal entry λ. Prove that the characteristic polynomial and minimal polynomial of $J_{\lambda,k}$ are both equal to $(x - \lambda)^k$.

   (b) Prove that $J_{\lambda,k}$ has a single one-dimensional eigenspace $E_\lambda = \langle e_1 \rangle$.

   (c) For any linear map T with eigenvalue λ, show that the geometric multiplicity of λ – the dimension of the eigenspace E_λ – is equal to the number of Jordan blocks with diagonal entry λ in the Jordan canonical form of T.

   (d) Let λ be an eigenvector of T. Define the *generalized eigenspace of* λ to be the subspace

   $$G_\lambda = \{v \mid (\lambda I - T)^k v = 0 \text{ for some integer } k > 0\}$$

   (e) Show (in a sentence) that $E_\lambda \subseteq G_\lambda$.

   (f) Show that the generalized eigenspace G_λ of V is precisely the direct sum of submodules of the form $\mathbb{F}[x]/(x - \lambda)^\alpha$ in the elementary divisor form of the decomposition of V (called the $(x - \lambda)$-*primary component* of V), by analyzing the action of x on this decomposition.

(g) Re-prove part (f), this time by analyzing the Jordan canonical form of T. Show that the generalized eigenspace for λ is precisely the subspace of V corresponding to the Jordan blocks with diagonal entry λ. Start by observing that a Jordan block $J_{\mu,k}$ will satisfy the polynomial $(\lambda I - J_{\mu,k})^m$ for some m if and only if $\mu = \lambda$.

(h) Conclude that V decomposes into a direct sum of generalized eigenspaces for T, and that the algebraic multiplicity of an eigenvalue λ is equal to sum of the sizes of the corresponding Jordan blocks, which is equal to the dimension of G_λ.

(i) Note as a corollary that dimension of the eigenspace E_λ is no greater than the algebraic multiplicity of λ. Under what conditions are they equal?

4. State instructions for how to read off the following data from the Jordan canonical form of a linear map T, and state each for the specific map T_0 given below. You do not need justify your computations or instructions.

$$T_0 = \begin{bmatrix} 2 & 1 & & & & & & & & \\ & 2 & 1 & & & & & & & \\ & & 2 & & & & & & & \\ & & & 2 & 1 & & & & & \\ & & & & 2 & 1 & & & & \\ & & & & & 2 & & & & \\ & & & & & & 2 & & & \\ & & & & & & & 5 & 1 & \\ & & & & & & & & 5 & \\ & & & & & & & & & 7 \end{bmatrix}$$

(a) The eigenvalues of T (with algebraic multiplicity).

(b) The determinant of T.

(c) The characteristic polynomial of T.

(d) The minimal polynomial of T.

(e) The elementary divisors of T.

(f) The invariant factors of T.

(g) The dimension of the eigenspace E_λ associated to any eigenvalue λ of T.

(h) The dimension of the generalized eigenspace G_λ for any eigenvalue λ of T.