Reading: Dummit–Foote Ch 11.3, 11.5, 18.1, 18.3
Other suggested reading: Fulton–Harris "Representation Theory: A first course", Ch 1–2.

Summary of definitions and main results

Definitions we've covered: V^G, dual space V^* of V, exterior powers, symmetric powers, character

Main results: Maschke's theorem; induced $\mathbb{F}[G]$–modules structures on $V \oplus W$, $\mathrm{Hom}_k(V, W)$, V^*, $V \otimes W$, $\wedge^k V$, $\mathrm{Sym}^k(V)$; orthogonality of characters

Warm-Up Questions

1. Given an example of a ring R and an R–module M that is:
   (a) irreducible
   (b) reducible, but not decomposable
   (c) decomposable, but not completely reducible
   (d) completely reducible, but not irreducible
2. Let V be a representation of a group G, and recall that V^G denotes the set of vectors in V that are fixed pointwise by the action of every group element $g \in G$. Verify that V^G is a linear subspace of V.
3. Let V and W be representations of a group G over a field k. Define the induced action of G on the k–vector space $\mathrm{Hom}_k(V, W)$, and verify that it satisfies the definition of a representation of G.
4. (a) Let $\mathbb{C}^n = \langle e_1, \ldots, e_n \rangle$ be the canonical representation of the symmetric group S_n by signed permutation matrices. Explicitly describe the action of the *averaing map* on $\mathbb{C}^n$:
   $$\begin{aligned} \psi_{av} : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ v &\longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot v \end{aligned}$$
   (b) Suppose v is an element of the standard subrepresentation $\underline{Std} = \{a_1 e_1 + \cdots + a_n e_n \mid \sum a_i = 0\}$. What is $\psi_{av}(v)$? *Hint:* First check $\psi_{av}(v)$ on the basis vectors $v = (e_1 - e_i)$ for $\underline{Std}$.
   (c) Interpret your answer to the previous question, given that we know $\psi_{av} : V \to V$ is a linear projection onto V^G.
5. Let V denote the standard representation of S_3 over $\mathbb{C}$.
   (a) Use character theory to show that $V \otimes V$ demcomposes into a sum of one copy of each of the trivial representation, the alternating representation, and V.
   (b) Let $\omega = e^{2\pi i/3}$. Verify that the vectors $\alpha = (\omega, 1, \omega^2)$ and $\beta = (1, \omega, \omega^2)$ form a basis for V as a subrepresentation of $\mathbb{C}^3$. Verify that
   $$V \otimes_{\mathbb{C}} V = \langle \alpha \otimes \alpha,\ \beta \otimes \beta \rangle \oplus \langle \alpha \otimes \beta + \beta \otimes \alpha \rangle \oplus \langle \alpha \otimes \beta - \beta \otimes \alpha \rangle$$
   gives a decomposition of $V \otimes_{\mathbb{C}} V$ into S_3–invariant subspaces. Identify each of these representations. (Hint: The elements $(1\ 2), (1\ 2\ 3)$ generate S_3, so it is enough to check that they stabilize the subspaces.)
6. Let R be a commutative ring and M an R–module. Verify that our explicit construction of the symmetric power $\mathrm{Sym}^k(M)$ and the exterior power $\Lambda^k M$ satisfy the stated universal properties.

7. Let G be a finite group and $\phi : G \to GL(V)$ a G–representation over a field $\mathbb{F}$ with character $\chi_V : G \to \mathbb{F}$. Prove that if V is 1-dimensional, then $\chi_V = \phi$. Show by example that if V is at least 2 dimensional, χ_V may not be a group homomorphism.

8. Let G be a finite group. Verify that the space of $\mathbb{C}$-valued class functions on G form a $\mathbb{C}$-vector space with dimension equal to the number of G conjugacy classes.

9. Let V be a finite dimensional vector space over $\mathbb{C}$. Recall that a *(Hermitian) inner product* on V is a function
$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$
satisfying the following properties:
   - (Conjugate symmetry)
   $$\langle x, y \rangle = \overline{\langle y, x \rangle} \qquad \forall x, y \in V$$
   - (Linearity in the first coordinate)
   $$\langle ax, y \rangle = a\langle x, y \rangle \qquad \text{and} \qquad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle \qquad \forall x, y, z \in V, \alpha \in \mathbb{C}$$
   - (Positive definiteness)
   $$\langle x, x \rangle \geq 0 \qquad \text{and} \qquad \langle x, x \rangle = 0 \Rightarrow x = 0 \qquad \forall x \in V$$

   (a) Suppose that there is set of vectors $e_1, e_2, \ldots, e_n$ in V that is *orthonormal* with respect to the inner product $\langle \cdot, \cdot \rangle$. This means
   $$\langle e_i, e_j \rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$
   Prove that these vectors are linearly independent, and therefore form a basis for the space they span.

   (b) Let $v = a_1e_1 + \cdots + a_ne_n$ be an element of V. Show that
   $$\langle v, e_i \rangle = a_i.$$

   (c) Show that
   $$\langle v, v \rangle = |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2.$$

   (d) Suppose you have a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ which you know satisfies the conjugate-symmetry and linearity properties of an inner product. Show that, if V has an basis that is orthonormal with respect to the function, then it must be positive definite.

   (e) Suppose that $v = a_1e_1 + \cdots + a_ne_n$ for **nonnegative integer** coefficients a_i. Show that
   $$\langle v, v \rangle = a_1^2 + a_2^2 + \cdots + a_n^2,$$
   and conclude that $\langle v, v \rangle = 1$ if and only if $v = e_i$ for some i.

Assignment Questions

1. (a) Let V be a finite dimensional vector space over a field k. Given a choice of basis $B = \{b_1, \ldots, b_n\}$ for V, show that there is a *dual basis* $B^* = \{b^1, \ldots, b^n\}$ for V^*, where b^i is the linear functional defined on the basis B by
   $$b^i(b_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$
   and extended k-linearly. Conclude that a choice of basis for V defines an isomorphism of vector spaces $V \cong V^*$, $b_i \mapsto b^i$.
   Remark: Although V and V^* are isomorphic as abstract vector spaces, they are not *naturally isomorphic* in the sense that any isomorphism involves a choice of basis.

(b) Let $\bullet$ denote the dot product on vectors in V written with respect to the basis B. Verify that for any $w = \sum_i c_i b_i \in V$

$$b^i(w) = b_i \bullet w = c_i.$$

Show more generally that the isomorphism from part (a) is in fact the map

$$\begin{aligned} V &\longrightarrow V^* \\ v &\longmapsto \{w \mapsto v \bullet w\} \end{aligned}$$

(c) Show that if $A : V \to V$ is a linear map (given by a matrix with respect to the basis B). Show that

$$Av \bullet w = v \bullet A^T w \qquad \text{and (since } (A^T)^T = A) \qquad v \bullet Aw = A^T v \bullet w$$

where A^T denotes the transpose of the matrix A.

(d) Suppose G is a group with a linear action on V given by $\rho : G \to GL(V)$. Let ρ^* denote the induced action of G on V^* given by

$$\Big(\rho^*(g)(\phi)\Big)(v) = \phi\Big(\rho(g)^{-1}(v)\Big) \qquad \forall\, \phi \in V^*,\ g \in G$$

If A is the matrix representing the action of a group element $g \in G$ on V with respect to the basis B, show that the matrix for g on V^* with respect to B^* is given by $(A^{-1})^T$, the inverse transpose of A.

(e) Conclude that if G is a finite group and $k = \mathbb{C}$, and if $\{\lambda_i\}$ are the eigenvalues for the action of an element g on V, then the eigenvalues for the action of g on V^* are $\{\lambda_i^{-1}\}$, and moreover for each i, $\lambda_i^{-1} = \overline{\lambda_i}$. (For this question, you can quote properties of the transpose without proof).

(f) Conclude the formula for the characters: $\chi_{V^*}(g) = \overline{\chi_V(g)}$.

2. Let V and W be finite-dimensional representations of a finite group G over a field $\mathbb{F}$.

(a) Suppose that $\mathbb{F}$ is algebraically closed, and that A and B are finite order (therefore diagonalizable) maps on vector spaces U and U'. Show that the trace of $A \otimes B$ on $V \otimes_{\mathbb{F}} W$ is the product $\text{Trace}(A)\text{Trace}(B)$. Conclude that the character $\chi_{V \otimes_{\mathbb{F}} W}(g) = \chi_V(g)\chi_W(g)$.
Remark: This result also holds when A and B are not diagonalizable, and can be proven (with a little more effort) by considering the bases for U and U' putting A and B into Jordan canonical form. It can also be proven even when the field is not algebraically closed, by extension of scalars to the algebraic closure.

(b) Let $\mathbb{F}$ be any field. Construct an isomorphism of G–representations $\text{Hom}_{\mathbb{F}}(V, W) \cong V^* \otimes_{\mathbb{F}} W$. This isomorphism should be *natural*, that is, it should not require a choice of basis for V or W.

(c) Suppose $\mathbb{F} = \mathbb{C}$. Show that the character of $\text{Hom}_{\mathbb{C}}(V, W)$ is

$$\chi_{\text{Hom}_{\mathbb{C}}(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g).$$

This will be a key result in our proof of orthogonality of characters!

3. Let $\mathbb{F}$ be a field, and V a vector space over $\mathbb{F}$ with basis $\{x_1, \ldots, x_n\}$.

(a) Verify that $\text{Sym}^k(V)$ is a vector space over $\mathbb{F}$ with basis given by the set of monomials in the variables $\{x_1, x_2, \ldots, x_n\}$ of total degree k. (*Remark:* There are $\binom{n+k+1}{n-1}$ such monomials).

(b) Verify that $\wedge^k V$ is isomorphic to the $\mathbb{F}$–vector space with a basis given by elements of the form $x_{i_1} x_{i_2} \cdots x_{i_k}$ with $i_1 < i_2 < \cdots < i_k$.
Hint for (a) and (b): To show these elements are linearly independent, is enough to use the universal property to define a symmetric or alternating multilinear map $V^k \to \mathbb{C}$ that factors through $\text{Sym}^k V$ or $\wedge^k V$ in such a way that it takes on the value 1 on one basis element and 0 on all others.

(c) Let R be a commutative ring and M and R–module. Show that the additive groups

$$T^*M := \bigoplus_{i=0}^{\infty} M^{\otimes i} \qquad \mathrm{Sym}^*M := \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(M) \qquad \wedge^* M := \bigoplus_{i=0}^{\infty} \wedge^i M$$

each have a natural ring structure. You do not need to check the axioms for a ring, but define and briefly describe the multiplication in each case. The multiplication on T^*M is called *noncommutative*, the multiplication on Sym^*M is *commutative*, and the multiplication on $\wedge^*M$ is called *anti-commutative*.

4. (a) Show that every permutation $\sigma \in S_n$ is conjugate to its inverse σ^{-1}.

   (b) Prove that every finite dimensional complex representation V of the symmetric group S_n is *self-dual*, that is, there is an isomorphism of S_n–representations $V \cong V^*$.
   Hint: We will prove on Wednesday that a finite-dimensional complex representation of a finite group is determined by its character.

5. (**Bonus Question:** The following question will require material from Wed 27 May's class.)

   (a) Compute the character table for the symmetric group S_5 over $\mathbb{C}$.

   (b) Let $\underline{\mathrm{Std}}$ denote the standard representation of S_5. Use the character table to find the decomposition of $\underline{\mathrm{Std}} \otimes_{\mathbb{C}} \underline{\mathrm{Std}}$ and $\wedge^3\underline{\mathrm{Std}}$ into irreducible S_5–representations.