# Final Exam

Math 122
June 2015
Jenny Wilson

Name: ______________________________

**Instructions:** This exam has 9 questions for a total of 50 points.

This exam is closed-book. No books, notes, cell phones, or calculators are permitted. Scratch paper is available.

Fully justify your answers unless directed otherwise. You may cite any results from class or the homeworks without proof, but do give a complete statement of the result you are using.

You have 180 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

| Question | Points | Score |
|---|---|---|
| 1 | 7 | |
| 2 | 4 | |
| 3 | 4 | |
| 4 | 9 | |
| 5 | 5 | |
| 6 | 5 | |
| 7 | 6 | |
| 8 | 4 | |
| 9 | 6 | |
| Total: | 50 | |

1. (a) (2 points) Classify (up to conjugacy) all linear maps $T : \mathbb{C}^4 \to \mathbb{C}^4$ with characteristic polynomial $c(x) = (x-1)^3(x-2)$. **No justification necessary.**

Each conjugacy class has a unique representative in Jordan canonical form. From the characteristic polynomial we see that $T$ has eigenvalue 1 with algebraic multiplicity 3 (JCF has three 1's on its diagonal), and eigenvalue 2 with algebraic multiplicity 1 (JCF has one 2 on its diagonal).

Without additional information, we do not know whether there are 1, 2 or 3 Jordan blocks associated to the eigenvalue 1. This gives the three possible Jordan form representatives:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Every linear transformation $T : \mathbb{C}^4 \to \mathbb{C}^4$ is conjugate to one and only one of these three matrices.

(b) (1 point) Write down the companion matrix (in the sense of rational canonical form) to the polynomial $x^4 + 3x^3 + x - 1$. **No justification necessary.**

In general, the companion matrix to a polynomial $p(x) = x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$ is the $(k \times k)$ matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & 0 & \cdots & 0 & -b_1 \\ 0 & 1 & 0 & \cdots & 0 & -b_2 \\ 0 & 0 & 1 & \cdots & 0 & -b_3 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -b_{k-1} \end{bmatrix}$$

(This is the matrix that represents the linear map "multiplication by $x$" on the $\mathbb{F}[x]$–module $\dfrac{\mathbb{F}[x]}{\langle p(x) \rangle}$ with respect to the basis $\{1, x, x^2, \ldots, x^{k-1}\}$. It has the property that its characteristic polynomial is $p(x)$.)

For the polynomial $x^4 + 3x^3 + x - 1$, the companion matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}.$$

(c) (4 points) Under each of the matrices $A$ (in rational canonical form) and $B$ (in Jordan canonical form), fill in requested data. **No justification necessary.** You do not need to expand / factor / simplify your polynomials.

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

| | $A$ | $B$ |
|---|---|---|
| characteristic polynomial: | $(x+1)(x^3-x)$ | $(x-1)^3(x-2)^2(x-3)^2$ |
| minimal polynomial: | $(x^3-x)$ | $(x-1)^2(x-2)^2(x-3)$ |
| elementary divisors: | $(x+1),\ (x+1),\ (x-1),\ x$ | $(x-1)^2,\ (x-1),\ (x-2)^2,\ (x-2)^2,\ (x-3),\ (x-3)$ |
| invariant factors: | $(x+1),\ (x^3-x)$ | $(x-1)^2(x-2)^2(x-3),\ (x-1)(x-3)$ |
| dimension of the eigenspace $E_\lambda$ for each eigenvalue $\lambda$: | $\dim(E_{-1})=2$, $\dim(E_1)=1$ $\dim(E_0)=1$ | $\dim(E_1)=2$, $\dim(E_2)=1$ $\dim(E_3)=2$ |
| dimension of the generalized eigenspace $G_\lambda$ for each eigenvalue $\lambda$: | $\dim(G_{-1})=2$, $\dim(G_1)=1$, $\dim(G_0)=1$ | $\dim(G_1)=3$, $\dim(G_2)=2$, $\dim(G_3)=2$ |

2. (4 points) Prove or disprove: There is an isomorphism of $\mathbb{Q}$ vector spaces $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

(A solution to this question appeared in a homework assignment – but give a complete proof here; do not just cite this assignment).

This statement is true, and we will construct an isomorphism of rational vector spaces. First, consider the map

$$\begin{aligned} \varphi : \mathbb{Q} \times \mathbb{Q} &\longrightarrow \mathbb{Q} \\ (a,b) &\longmapsto ab \end{aligned}$$

We can check that this map is $\mathbb{Z}$–balanced: for all $a, b, c \in \mathbb{Q}$ and $n \in \mathbb{Z}$,

$$\begin{aligned} \varphi(a+c,b) = (a+c)(b) = ab + cb &= \varphi(a,b) + \varphi(c,b) \\ \varphi(a,b+c) = (a)(b+c) = ab + ac &= \varphi(a,b) + \varphi(a,c) \\ \varphi(an,b) = (an)(b) = a(nb) &= \varphi(a,nb) \end{aligned}$$

It follows that $\varphi$ factors through the tensor product, giving a map

$$\begin{aligned} \Phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} &\longrightarrow \mathbb{Q} \\ a \otimes b &\longmapsto ab \end{aligned}$$

Moreover, this map is a map of rational vector spaces, since for any $q \in \mathbb{Q}$ we have $\Phi(qa \otimes b) = qab = q\Phi(a \otimes b)$.

To show $\Phi$ is an isomorphism, we will construct an inverse. Define

$$\begin{aligned} \Psi : \mathbb{Q} &\longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \\ q &\longmapsto q \otimes 1 \end{aligned}$$

Then

$$\Phi \circ \Psi(q) = \Phi(q \otimes 1) = (q)(1) = q,$$

so $\Phi \circ \Psi$ is the identity map on $\mathbb{Q}$. In the other direction,

$$\Psi \circ \Phi(a \otimes b) = \Psi(ab) = (ab) \otimes 1$$

To complete the proof that $\Psi \circ \Phi$ is the identity map on $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, we need to show that $(ab) \otimes 1 = a \otimes b$. Write $a$ and $b$ as reduced integer fractions $a = \frac{m}{n}$, $b = \frac{k}{\ell}$. Using the property that $qn \otimes p = q \otimes np$ for any integer $n$, we have:

$$\begin{aligned} (ab) \otimes 1 &= \left(\frac{m}{n}\right)\left(\frac{k}{\ell}\right) \otimes 1 = \left(\frac{m}{n}\right)\left(\frac{k}{\ell}\right) \otimes \frac{\ell}{\ell} = \left(\frac{m}{n}\right)\left(\frac{1}{\ell}\right) k \otimes \ell \left(\frac{1}{\ell}\right) \\ &= \left(\frac{m}{n}\right)\left(\frac{1}{\ell}\right) \ell \otimes k \left(\frac{1}{\ell}\right) = \left(\frac{m}{n}\right)\left(\frac{\ell}{\ell}\right) \otimes \frac{k}{\ell} = a \otimes b. \end{aligned}$$

So we have $\Psi = \Phi^{-1}$, and we conclude that $\Phi$ is an isomorphism of $\mathbb{Q}$ vector spaces $\Phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$.

3. (4 points) Find an example of a ring $R$ and a right $R$–module $M$ such that the functor from $R$–modules to abelian groups $M \otimes_R -$ is **not** exact. Fully explain your solution.

To show that $M \otimes_R -$ is not exact, we need to find a short exact sequence of $R$–modules that, after applying the functor $M \otimes_R -$, yields a short exact sequence of abelian groups that is not exact.

One solution: Let $R = \mathbb{Z}$, and let $M = \mathbb{Z}/2\mathbb{Z}$. Consider the short exact sequence of $\mathbb{Z}$–modules

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{(\text{mod } 2)} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

Here, the first map is multiplication by 2, and the second map is reduction mod 2.

Applying the functor, we get:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$a \otimes b \longmapsto 2a \otimes b, \quad c \otimes d \longmapsto (c \text{ mod } 2) \otimes d$$

We know from the homework that we have an isomorphism of $\mathbb{Z}$–modules

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} &\xrightarrow{\cong} \mathbb{Z}/\gcd(m,n)\mathbb{Z} \\ a \otimes b &\longmapsto ab \pmod{\gcd(m,n)} \\ d \otimes 1 &\longleftarrow d \pmod{\gcd(m,n)} \end{aligned}$$

Under this isomorphism, our two maps become

$$d \xrightarrow{\cong} d \otimes 1 \xrightarrow{2} 2d \otimes 1 \xrightarrow{\cong} 2d \qquad \text{and} \qquad d \xrightarrow{\cong} d \otimes 1 \xrightarrow{(\text{mod } 2)} (d \text{ mod } 2) \otimes 1 \xrightarrow{\cong} (d \text{ mod } 2)$$

The resultant short exact sequence is:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\text{mod } 2)} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$$d \longmapsto 2d, \quad c \longmapsto (c \text{ mod } 2)$$

But multiplication by 2 is the zero map on $\mathbb{Z}/2\mathbb{Z}$, and reduction modulo 2 is the identity map. The sequence is:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{id} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

In particular the zero map $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is not injective, so this sequence fails to be exact.

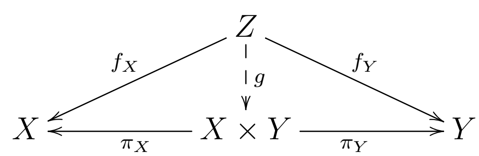
4. (a) (1 point) Let $R$ be a ring, and let $X$ and $Y$ be $R$–modules. Define the *direct product* $X \times Y$.

The direct product $X \times Y$ is defined as the set $\{(x, y) \mid x \in X, \ y \in Y\}$ with $R$–module structure

$r(x_1, y_1)+(x_2, y_2) := (rx_1+x_2, ry_1+y_2)$ for all $x_1, x_2 \in X$, $y_1, y_2 \in Y$ and $r \in R$.

(Since it is a finite direct product, it is isomorphic to the (external) direct sum $X \oplus Y$.)

(b) (4 points) Prove that the product (along with the natural projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$) satisfies the following universal property:
For any $R$–module $Z$ and $R$–module maps $f_X : Z \to X$ and $f_Y : Z \to Y$, there is a unique $R$–module map $g : Z \to X \times Y$ making the following diagram commute.




Suppose that $Z$ is an $R$–module with maps

$$f_X : Z \to X \qquad \text{and} \qquad f_Y : Z \to Y.$$

Then define a map

$$\begin{aligned} g : Z &\longrightarrow X \times Y \\ z &\longmapsto (f_X(z), f_Y(z)) \end{aligned}$$

We can verify that $g$ is a map of $R$–modules. Given any $r \in R$ and $z_1, z_2 \in Z$,

$$\begin{aligned} g(rz_1 + z_2) &= \Big(f_X(rz_1 + z_2), f_Y(rz_1 + z_2)\Big) \\ &= \Big(rf_X(z_1) + f_X(z_2), rf_Y(z_1) + rf_Y(z_2)\Big) \\ &= r\Big(f_X(z_1), f_Y(z_1)\Big) + \Big(f_X(z_2), f_Y(z_2)\Big) \\ &= rg(z_1) + g(z_2) \end{aligned}$$

Moreover, we can verify that $g$ makes the diagram commute:

$$\pi_X \circ g(z) = \pi_X\Big(f_X(z), f_Y(z)\Big) = f_X(z)$$

and similarly $\pi_Y \circ g(z) = f_Y(z)$. This establishes the existence of the map $g$.

To prove uniqueness, let $g'$ be any map the completes the diagram. For $z \in Z$, and suppose $g'(z) = (x, y)$ for some $x \in X$ and $y \in Y$. The commutativity of the diagram means we must have

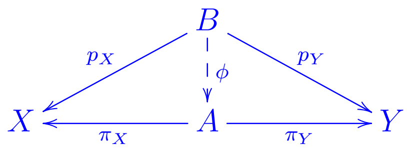
$$f_X(z) = \phi_X \circ g'(z) = \pi_X(x, y) = x \qquad \text{and } f_Y(z) = \phi_Y \circ g'(z) = \pi_Y(x, y) = y$$

so $(x, y) = \Big(f_X(z), f_Y(z)\Big)$. This shows that $g' = g$, and we conclude that $g$ is the unique map of $R$–modules making the diagram commute.

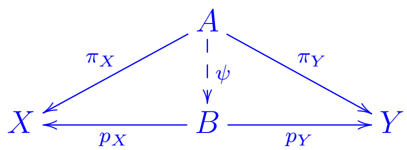
(c) (4 points) Show that this universal property defines the product $X \times Y$ (with its projection maps) uniquely up to unique isomorphism.

Suppose there were two $R$–modules satisfying this universal property: The $R$–module $A$, along with projection maps $\pi_X$ and $\pi_Y$, and the $R$–module $B$, along with projection maps $p_X$ and $p_Y$.

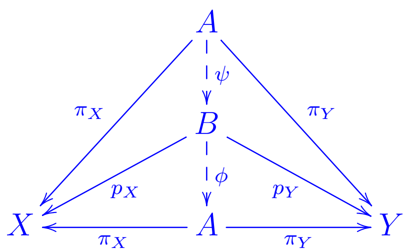
Then by the universal property for $A$, there is a unique map of $R$–modules $\phi$ making the following diagram commute:

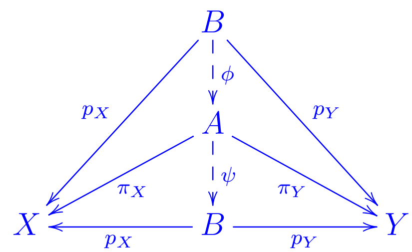



Similarly, by the universal property for $B$¡ there is a unique map of $R$–modules $\psi$ making the following diagram commute:

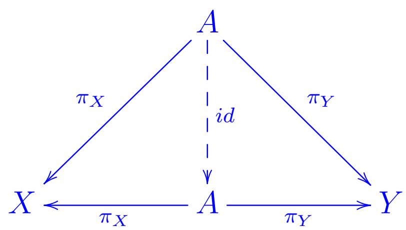



We can combine these two diagrams into two larger commutative diagrams:

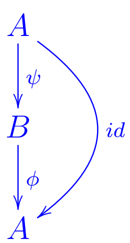


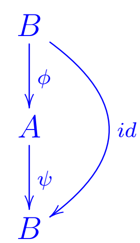

Consider the left-hand diagram (call it $L$). It is clear that the identity map makes the following diagram commute:



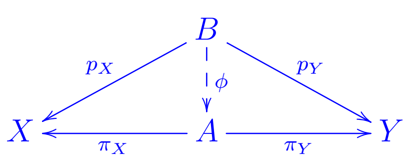

so again citing the universal property, by uniqueness of the maps completing the digram $L$, we must have $\phi \circ \psi = id$:

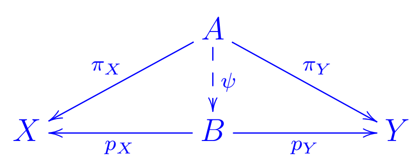



and by the same argument, from the right-hand diagram




Then $\phi \circ \psi = id$ and $\psi \circ \phi = id$. The maps $\phi$ and $\psi$ are isomorphisms between $A$ and $B$, and they are the unique isomorphisms compatible with the projection maps, in the sense that they are the unique maps making the diagrams commute:




We conclude that the data of the direct product $X \times Y$ along with its projection maps $\pi_X$ and $\pi_Y$ are uniquely determined up to unique isomorphism.

5. (a) (2 points) An matrix $N$ is called *nilpotent* if $N^k = 0$ for some positive integer $k$. Prove that if $N$ is an $n \times n$ nilpotent matrix, then $N^n = 0$.

Suppose an $n \times n$ matrix $N$ satisfies $N^k = 0$ for some positive integer $k$. The minimal polynomial $m_N(x)$ of $N$ is the principal generator of the ideal of polynomials that annihilate $N$, so $m_N(x)$ must divide $x^k$, which means that $m_N(x)$ must be a power of $x$.

We proved that the degrees of the invariant factors of $N$ must sum to $n$, so in particular $m_N(x)$ has degree at most $n$, and it must divide $x^n$. Since $N$ satisfies its minimal polynomial, it must satisfy the polynomial $x^n$.

(b) (3 points) Prove that any $n \times n$ matrix over $\mathbb{C}$ can be written as the sum of a nilpotent matrix and a diagonalizable matrix.

Let $J_{\lambda,k}$ denote the $(k \times k)$ Jordan block matrix with diagonal entry $\lambda$:

$$J_{\lambda,k} = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

Then an $n \times n$ matrix $A$ is conjugate (via some matrix $B$) to a matrix in Jordan canonical form. But we can decompose each block $J_{\lambda,k}$ into the sum $\lambda I_k + J_{0,k}$, where $I_k$ is the $k \times k$ identity matrix. Thus $BAB^{-1}$ is equal to

$$\begin{bmatrix} J_{\lambda_1,k_1} & 0 & \cdots & 0 \\ 0 & J_{\lambda_2,k_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & J_{\lambda_N,k_N} \end{bmatrix} = \begin{bmatrix} \lambda_1 I_{k_1} & 0 & \cdots & 0 \\ 0 & \lambda_2 I_{k_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_N I_{k_N} \end{bmatrix} + \begin{bmatrix} J_{0,k_1} & 0 & \cdots & 0 \\ 0 & J_{0,k_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & J_{0,k_N} \end{bmatrix}$$

Call the first matrix in this sum $D$, and the second matrix $N$, so we have

$$BAB^{-1} = D + N$$
$$A = B^{-1}(D+N)B = B^{-1}DB + B^{-1}NB$$

Since $D$ is a diagonal matrix, its conjugate $B^{-1}DB$ is by definition diagonalizable. Moreover, it was a result on the Homework that $J_{0,k}$ has minimal polynomial $x^k = 0$. This means $N^k = 0$, where $k$ is the size of the largest Jordan block. Since conjugation is a ring automorphism, it follows that $(B^{-1}NB)^k = B^{-1}N^kB = 0$, so we see that the matrix $B^{-1}NB$ is nilpotent.

The desired decomposition is: $A = B^{-1}DB + B^{-1}NB$.

6. (5 points) Let $R$ be a PID that is **not** a field. Suppose that a nonzero $R$–module $M$ is *divisible*, that is, for each $m \in M$ and nonzero $r \in R$, there is some $m' \in M$ such that $rm' = m$. Show that $M$ is not finitely generated.

We proceed by contradiction: Assume $M$ is a finitely generated module over a PID $R$ that is not a field, and that $M$ is divisible.

According to the invariant factor form of the structure theorem for finitely generated modules over a PID, the module $M$ decomposes as

$$M \cong R^s \oplus \frac{R}{\langle a_1 \rangle} \oplus \frac{R}{\langle a_2 \rangle} \oplus \cdots \oplus \frac{R}{\langle a_k \rangle}$$

for some invariant factors $a_1 | a_2 | \cdots | a_k \in R$. We have seen in class that $a_k$ generates the annihilator of the torsion submodule $\text{Tor}(M)$.

Let $\{b_1, b_2, \ldots, b_s\}$ be a basis for the free factor $R^s$ (if $s = 0$ this is the empty set), and let $b_{i+s}$ be the image of 1 in $\frac{R}{\langle a_i \rangle}$. The elements $b_1, \ldots, b_{i+k}$ are a generating set for $M$.

First suppose that $M$ has positive rank. Let $m = b_1$, and let $r$ be any nonzero element of $R$. Since $M$ is divisible, there is some $m' = \sum_{i=1}^{s+k} r_i b_i \in M$ with $rm' = m = b_1$. Multiplying both sides by $a_k$ annihilates the torsion elements $b_{s+1}, \ldots, b_{s+k}$ in the sum $m'$, so

$$a_k r m' = a_k(r)(r_1 b_1 + \ldots r_s b_s) = a_k b_1$$

Substacting $a_k b_1$ from both sides:

$$(a_k r r_1 - a_k) b_1 + a_k r r_2 b_2 + \cdots a_k r r_s b_s = 0$$

But by definition of basis, this linear combination can be zero only if each coefficient is zero, in particular, $(a_k r r_1 - a_k) = 0$. Then $a_k(rr_1 - 1) = 0$, and since $R$ is a domain, $rr_1 = 1$. We have constructed an inverse to an arbitrary nonzero element $r \in R$, so we conclude that $R$ is a field, a contradiction.

Alternatively, suppose that $s = 0$, so $M$ is torsion. Then let $m = b_{i+k}$ and let $r = a_k$. Since $M$ is divisible, there is some $m'$ in $M$ with $a_k m' = m = b_{i+k}$. But $a_k$ annihilates $M$, so this is a contradiction.

In all cases, we have reached a contradiction. We conclude that if $M$ is a nonzero divisible $R$–module over a PID, either $R$ is a field or $M$ is not finitely generated.



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7. (a) (1 point) Let $R$ be a ring, and $M$ an $R$–module. State what it means for $M$ to be a *Noetherian* $R$–module.

An $R$–module $M$ is Noetherian if it satisfies the ascending chain condition: any increasing chain of submodules

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M,$$

stabilizes. This means there is some index $m$ such that $M_k = M_m$ for all $k \geq m$.

We proved in class that this definition is equivalent to the following: an $R$–modules $M$ is Noetherian if every $R$–submodule of $M$ is finitely generated.

(b) (5 points) Let $R$ be a ring, and let

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\varphi} Q \longrightarrow 0$$

be a short exact sequence of $R$–modules. Show that if $K$ and $Q$ are Noetherian $R$–modules, then $M$ is Noetherian.

This problem is very closely related to Homework #8 Question 1.

Given the short exact sequence

$$0 \longrightarrow K \longrightarrow M \xrightarrow{\varphi} Q \longrightarrow 0$$

suppose that $K$ and $Q$ are both Noetherian modules. Let $N$ be any $R$–submodule of $M$; to show that $N$ is Noetherian is suffices to show that $N$ is finitely generated. Consider the restriction of $\varphi$ to $N$. An element is in the kernel of this map precisely when it is in both $N$ and $K$. This gives a short exact sequence

$$0 \longrightarrow K \cap N \longrightarrow N \xrightarrow{\varphi|_N} \varphi(N) \longrightarrow 0$$

The $R$–module $K \cap N$ is a submodule of $K$, and $\varphi(N)$ is a submodule of $Q$, so our assumption that $K$ and $Q$ are Noetherian implies that $K \cap N$ and $\varphi(N)$ are finitely generated.

Let $x_1, x_2, \ldots, x_k$ be a finite generating set for $K \cap N$, and let $z_1, z_2, \ldots, z_m$ be a finite generating set for the quotient $\phi(N)$. Since $\phi$ surjects we can choose a finite set $y_1, \ldots, y_m \in N$, with $y_i$ a preimage of $z_i$. We claim that $x_1, \ldots, x_k, y_1, \ldots, y_m$ is a generating set fo $N$.

To see this let $n \in N$ be any element. Then $\phi(n) \in \phi(N)$ is of the form

$$\phi(n) = r_1 z_1 + r_2 z_2 + \cdots + r_m z_m$$

for some coefficients $r_i \in R$. Take the preimage $r_1 y_1 + \cdots + r_m y_m$. It may not be the case that this preimage is equal to $n$, but we have

$$\begin{aligned}\phi\big(n - (r_1 y_1 + r_2 y_2 + \cdots + r_m y_m)\big) &= \phi(n) - \phi(r_1 y_1 + r_2 y_2 + \cdots + r_m y_m) \\ &= \phi(n) - (r_1 z_1 + r_2 z_2 + \cdots + r_m z_m) \\ &= 0\end{aligned}$$

So $n - (r_1 y_1 + r_2 y_2 + \cdots + r_m y_m) \in \ker(\phi) = K \cap N$. But this means that this element is in the span of the elements $x_i$, say, it is equal to $(s_1 x_1 + s_2 x_2 + \cdots + s_k x_k)$ for some $s_i \in R$. Putting this together, we get

$$n = (r_1 y_1 + r_2 y_2 + \cdots + r_m y_m) + (s_1 x_1 + s_2 x_2 + \cdots + s_k x_k)$$

and $n$ is in the span of the generating set $B = \{y_1, \ldots, y_m, x_1, \ldots, x_k\}$ as desired. We conclude that $B$ generates $N$, and so $N$ is finitely generated, and $M$ is Noetherian.



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8. (4 points) Give an example of any group $G$ and a complex representation of $G$ that is reducible but not decomposable. Justify your answer.

By Maschke's theorem, any such group must be infinite. We can use an example from class, when $G = \mathbb{Z}$.

Consider the representation defined by the group homomorphism

$$\phi : \mathbb{Z} \longrightarrow GL(\mathbb{C}^2) \qquad \mathbb{C}^2 = \langle e_1, e_2 \rangle$$
$$1 \longmapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Since 1 generates $\mathbb{Z}$, this extends uniquely to a map of abelian groups, the map

$$n \longmapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

Each of these matrices has an eigenvector $e_1$, which implies that the eigenspace $E_1 = \langle e_1 \rangle$ is stable under the action of the group; it is a $\mathbb{Z}$-subrepresentation.

However, $E_1$ has no direct complement in $\mathbb{C}^2$ that is stable under the action of $\mathbb{Z}$. To see this, consider the matrix $\phi(1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. A direct complement to $E_1$ would have to be a 1-dimensional subspace stable under the action of this matrix – that is, a second eigenspace. But this matrix is not diagonalizable: it is already in Jordan canonical form, and we know the Jordan canonical form of a matrix is unique, so it cannot be conjugate to a diagonal matrix.

Thus this representation $\mathbb{C}^2$ is not irreducible – it has a nontrivial subrepresentation – but it cannot be decomposed into a direct sum of subrepresentations.

9. Let $G$ and $H$ be finite groups with complex representations $V$ and $U$, respectively. We can construct a representation $V \otimes_{\mathbb{C}} U$ of $G \times H$ with action

$$(g, h)(v \otimes u) = gv \otimes hu$$

(a) (3 points) Compute the character of the $(G \times H)$–representation $V \otimes_{\mathbb{C}} U$ in terms of the characters $\chi_V$ and $\chi_U$ of $V$ and $U$.

Let $(g, h)$ be an element of $G \times H$. Since $G$ and $H$ are finite groups, by a result on the homework $g$ and $h$ must act on $V$ and $W$, respectively, by diagonalizable linear maps. Let $x_1, x_2, \ldots, x_n$ be a set of eigenvevtors for the action of $g$ on $V$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ , and let $y_1, y_2, \ldots y_m$ be eigenvectors for the action of $h$ on $U$ with eigenvalues $\mu_1, \mu_2, \ldots, \mu_m$. Then

$$(g, h)x_i \otimes y_j = g(x_i) \otimes h(y_j) = \lambda_i x_i \otimes \mu_j y_j = (\lambda_i \mu_j) x_i \otimes y_j$$

so we see that $x_i \otimes y_j$ is an eigenbasis for $V \otimes_{\mathbb{C}} U$ under the action of $(g, h)$ with associated eigenvalues $\lambda_i \mu_j$.

The trace of the action of $(g, h)$ is the sum of the eigenvalues:

$$\chi_{V \otimes_{\mathbb{C}} U}(g, h) = \sum_{i=1, j=1}^{n,m} \lambda_i \mu_j = \left( \sum_{i=1}^{n} \lambda_i \right) \left( \sum_{j=1}^{m} \mu_j \right) = \chi_V(g) \chi_U(h)$$

We conclude that $\chi_{V \otimes_{\mathbb{C}} U}(g, h) = \chi_V(g) \chi_U(h)$ for all $g \in G$ and $h \in H$.

(b) (3 points) Prove that if $V$ and $U$ are irreducible $G$ and $H$ representations (respectively), then $V \otimes_{\mathbb{C}} U$ is an irreducible $(G \times H)$–representation.

We proved in class that a representation of a group $W$ is irreducible if and only if its character satisfies $(\chi_W, \chi_W) = 1$. This implies that $(\chi_V, \chi_V)_G = 1$ and $(\chi_U, \chi_U)_H = 1$.

It suffices to compute $(\chi_{V\otimes_{\mathbb{C}}U}, \chi_{V\otimes_{\mathbb{C}}U})_{G\times H}$:

$$(\chi_{V\otimes_{\mathbb{C}}U}, \chi_{V\otimes_{\mathbb{C}}U})_{G\times H} = \frac{1}{|G \times H|} \sum_{(g,h)\in G\times H} \chi_{V\otimes_{\mathbb{C}}U}(g,h)\overline{\chi_{V\otimes_{\mathbb{C}}U}(g,h)}$$

We know $|G \times H| = |G||H|$. Moreover, by part (a), the character $\chi_{V\otimes_{\mathbb{C}}U}(g,h) = \chi_V(g)\chi_U(h)$. This gives:

$$\begin{aligned}
&= \frac{1}{|G||H|} \sum_{(g,h)\in G\times H} \chi_V(g)\chi_U(h)\overline{\chi_V(g)\chi_U(h)} \\
&= \frac{1}{|G||H|} \sum_{g\in G}\sum_{h\in H} \chi_V(g)\chi_U(h)\overline{\chi_V(g)\chi_U(h)} \\
&= \frac{1}{|G||H|} \sum_{g\in G}\sum_{h\in H} \chi_V(g)\overline{\chi_V(g)}\chi_U(h)\overline{\chi_U(h)} \\
&= \left(\frac{1}{|G|}\sum_{g\in G} \chi_V(g)\overline{\chi_V(g)}\right)\left(\frac{1}{|H|}\sum_{h\in H} \chi_U(h)\overline{\chi_U(h)}\right) \\
&= (\chi_V, \chi_V)_G(\chi_U, \chi_U)_H \\
&= (1)(1) \\
&= 1
\end{aligned}$$

This implies that the $(G \times H)$–representation $V \otimes_{\mathbb{C}} U$ is irreducible, as desired.

**Remark:** As an exercise, verify that the number of conjugacy classes in $G \times H$ is equal to the product $nm$ of the number $n$ of the conjugacy classes in $G$ and the number $m$ in $H$. Then, by doing a character computation like the one above, you can check that the $(G \times H)$–representation $V' \otimes_{\mathbb{C}} U'$ is nonisomorphic to $V \otimes_{\mathbb{C}} U$ if $U$ is not isomorphic to $U'$ and/or $V$ is not isomorphic to $V'$. Hence the set

$$\{V \otimes_{\mathbb{C}} U \mid V \text{ an irreducible } G\text{–representation, } U \text{ an irreducible } H\text{–representation } \}$$

is a set of $nm$ non-isomorphic $(G \times H)$–representations. By a cardinality argument, this must be a complete list of all the irreducible $(G \times H)$–representations.