Midterm Exam

Math 122
6 May 2015
Jenny Wilson

Name: ____

Instructions: This exam has 3 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless directed otherwise. You may cite any results from class or the homeworks without proof, but do give a complete statement of the result you are using.

You have 50 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	8	
2	6	
3	6	
Total:	20	

1. Compute the following. No justification needed.

   (a) (2 points) Describe the set of $\mathbb{C}[x]$–submodules **of complex dimension two** of the $\mathbb{C}[x]$–module $V \cong \mathbb{C}^3$, where x acts by the matrix $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ with respect to the standard basis e_1, e_2, e_3.

   $\mathbb{C}[x]$–submodules of V are precisely the vector subspaces U that are stable under the action of x, that is, vector subspaces U such that $xU \subseteq U$. Here, x stabilizes the span of the eigenvector e_3, and x stabilizes any vector in the 2–dimensional eigenspace $\langle e_1, e_2 \rangle$. The 2–dimensional stable subspaces of a diagonalizable matrix must be a sum of 1–dimensional stable subspaces (Exercise: Why?)

   Solution. The two dimensional $\mathbb{C}[x]$–submodules are the subspaces

   $$\langle\, e_1,\ e_2 \,\rangle \qquad \text{and} \qquad \langle\, ae_1 + be_2\ ,\ e_3 \,\rangle \text{ for any } a, b \in \mathbb{C},\ \ |a+b| = 1\ .$$

   (b) (2 points) Suppose that V is a 2–dimensional complex vector space, and $T : V \to V$ is a diagonalizable linear transformation with eigenvalues λ_1, λ_2. Suppose that W is a 3–dimensional complex vector space, and $R : W \to W$ a diagonalizable linear transformation with eigenvalues μ_1, μ_2, μ_3.
   List the eigenvalues of the linear map $T \otimes R : V \otimes_{\mathbb{C}} W \to V \otimes_{\mathbb{C}} W$.

   If we let e_1, e_2 be an eigenbasis for T, and f_1, f_2, f_3 be an eigenbasis for R, then $\{e_i \otimes f_j \mid i = 1, 2,\ j = 1, 2, 3\}$ will be an eigenbasis for R, since (using the definition of $T \otimes R$ and middle-linearity of tensors):

   $$(T \otimes R)(e_i \otimes f_j) = T(e_i) \otimes R(f_j) = (\lambda_i e_i) \otimes (\mu_j e_j) = (\lambda_i \mu_j)(e_i \otimes f_j).$$

   We see that $e_i \otimes f_j$ is an eigenvector with eigenvalue $\lambda_i \mu_j$.

   Solution. The six eigenvalues are:

   $$\lambda_1\mu_1, \quad \lambda_1\mu_2, \quad \lambda_1\mu_3, \quad \lambda_2\mu_1, \quad \lambda_2\mu_2, \quad \lambda_2\mu_3.$$

(c) (2 points) Simplify to a direct sum of abelian groups, without tensors:

$$(\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Q}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z}).$$

Since the tensor product is associative, this product may be expanded in any order. Because the tensor product distributes over direct sums, we have:

$$\Big((\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Q})\Big) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z})$$
$$= \Big((\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/15\mathbb{Z}) \oplus (\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/15\mathbb{Z}) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q})\Big) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z})$$

But we know from class and homework that:

- $(\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$ for all $m, n \in \mathbb{Z}_{>1}$
- $(\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}) \cong 0$ for all $n \in \mathbb{Z}_{>1}$
- $(\mathbb{Z} \otimes_{\mathbb{Z}} M) \cong M$ for any abelian group M.

and so our calculation simplifies:

$$\Big((\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/15\mathbb{Z}) \oplus (\mathbb{Z}/10\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/15\mathbb{Z}) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q})\Big) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z})$$
$$= \Big(\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Q}\Big) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Z})$$
$$= (\mathbb{Z}/5\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Z}/15\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus (\mathbb{Z}/15\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z})$$

We further know from the homework that:

- $(\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}) \cong 0$ for all $n \in \mathbb{Z}_{>1}$
- $(\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \cong 0$

so we have

$$(\mathbb{Z}/5\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Z}/15\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus (\mathbb{Z}/15\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) \oplus (\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z})$$
$$= \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Q}$$

Solution. $\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Q}$

(d) (2 points) List the elementary divisors and the invariant factors of the $\mathbb{Z}$–module

$$\mathbb{Z}^{13} \oplus \mathbb{Z}/20\mathbb{Z} \oplus \mathbb{Z}/50\mathbb{Z}.$$

By two applications of the Chinese Remainder Theorem:

$$\mathbb{Z}/20\mathbb{Z} \oplus \mathbb{Z}/50\mathbb{Z} \;\cong\; \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/25\mathbb{Z} \;\cong\; \mathbb{Z}/100\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$$

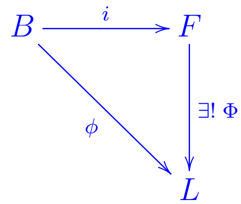
Solution.

Elementary divisors: $2, 4, 5, 25.$

Invariant factors: $10, 100.$

2. (a) (2 points) State the universal property of a free R–module F with basis B.

Solution. Let $i : B \hookrightarrow F$ denote the inclusion of the basis B. Then the free module F satisfies the following universal property: Given any R–module L and map of sets $\phi : B \to L$, there is a unique R–module homomorphism $\Phi : F \to L$ such that $\Phi \circ i = \psi$, that is, a unique R–module map making the following diagram commute:

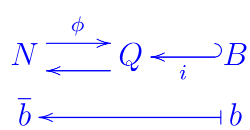


(b) (4 points) Consider a short exact sequence of R–modules

$$0 \longrightarrow M \xrightarrow{\psi} N \xrightarrow{\phi} Q \longrightarrow 0$$

Prove that, if Q is a free R–module, the sequence splits.

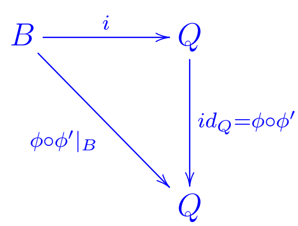
Solution. The Splitting Lemma states that the sequence splits if and only if we can find an R–module map $\phi' : Q \to L$ such that $\phi \circ \phi'$ is the identity map on Q. We will use the universal property of a free module to construct this map.
Let B be a basis for Q. Since ϕ surjects, for every $b \in B$ we can choose a preimage $\overline{b} \in N$. Then the universal property of the free module implies that the map of sets



extends uniquely to a map of R-modules $\phi' : Q \to N$.
The composite $\phi \circ \phi'$ is (by design) the identity map on B: for all $b \in B$,

$$\phi(\phi'(b)) = \phi(\overline{b}) = b.$$

Then, since the identity map id_Q completes the diagram below, the universal property's uniqueness provision implies that $\phi \circ \phi'$ must be the identity map on Q.



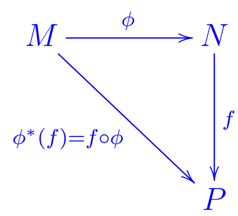
We have constructed a splitting homomorphism $\phi' : Q \to N$, and we conclude that the short exact sequence splits.

3. (a) (1 point) Let M, N, P be R–modules, and $\phi : M \to N$ an R–module homomorphism. Define the induced map:

$$\phi^* : \mathrm{Hom}_R(N, P) \to \mathrm{Hom}_R(M, P)$$

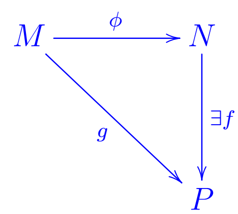
Solution. For any R–module map $f \in \mathrm{Hom}_R(N, P)$, we define

$$\phi^*(f) = f \circ \phi \in \mathrm{Hom}_R(M, P).$$



(b) (2 points) Describe what it would mean, in terms of the maps $\mathrm{Hom}_R(M, P)$, for ϕ^* to surject.

Solution. The map $\phi^* : \mathrm{Hom}_R(N, P) \to \mathrm{Hom}_R(M, P)$ will surject precisely if every R–module map $g : M \to P$ factors through the map $\phi : M \to N$, that is, if there exists some map $f : N \to P$ completing the following commutative diagram.



Then $g = f \circ \phi = \phi^*(f)$.

Remark: Sometimes (especially when ϕ is an inclusion), the map f is called an *extension* of g to N.

(c) (3 points) Give an example of $\mathbb{Z}$–modules M, N, P and an **injective** map of groups

$$\phi : M \to N$$

so that the induced map

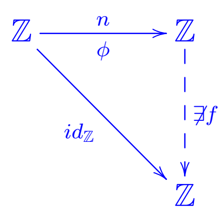
$$\phi^* : \mathrm{Hom}_{\mathbb{Z}}(N, P) \to \mathrm{Hom}_{\mathbb{Z}}(M, P)$$

does **not** surject.

(This proves that the functor $\mathrm{Hom}_{\mathbb{Z}}(-, P)$ is not exact.)

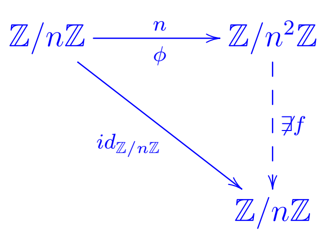
By part (b) it suffices to find an injective map $\phi : M \to N$ and a map $M \to P$ that does not factor through ϕ.

Solution 1. Let $M = N = P = \mathbb{Z}$, and let $n \in \mathbb{Z}_{>1}$. Then the identity map $id_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$ cannot factor through the injective map ϕ, "multiplication by n":



Any group map $f : \mathbb{Z} \to \mathbb{Z}$ completing the diagram would have to map $n \in \mathbb{Z}$ to $1 \in \mathbb{Z}$, and so $f(1)$ would have to satisfy $1 = f(n) = nf(1)$, an equation with no solutions in $\mathbb{Z}$.

Solution 2. Let $n > 1$ be an integer. Let $M = P = \mathbb{Z}/n\mathbb{Z}$ and $N = \mathbb{Z}/n^2\mathbb{Z}$. Then the identity map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ does not factor through the inclusion $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z}$ that maps $1 \pmod{n}$ to $n \pmod{n^2}$:



Any map $f : \mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ must take $n \pmod{n^2}$ to $f(n) = nf(1) \equiv 0 \pmod{n}$, and so the image of $\mathbb{Z}/n\mathbb{Z}$ is in the kernel of f. The map $f \circ n$ is the zero map, not the identity.

Other Solution Outlines. The identity map $\mathbb{Z} \to \mathbb{Z}$ does not factor through the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$, as $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. The identity map $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ does not factor through the inclusion $\mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$, as $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$.