1. State and prove the first isomorphism theorem for R–modules.

2. Let M and N be R–modules, and I an ideal of R contained in $\mathrm{ann}(M)$ and $\mathrm{ann}(N)$. Show that any map of R–modules $\phi : M \to N$ is also a map of (R/I)–modules. Conclude that $\mathrm{Hom}_R(M, N) = \mathrm{Hom}_{R/I}(M, N)$.

3. (a) If $a \in R$, prove that $Ra \cong R/\mathrm{ann}(a)$, where $\mathrm{ann}(a)$ denotes the annihilator of the left ideal generated by a.

   (b) Let M be an R–module. For $a, b \in M$, let $A = \{a, b\}$. Prove or disprove: $RA \cong R/I$, where I is the annihilator of the submodule generated by a and b.

4. (a) **(Chinese Remainder Theorem)** Let R be any ring, and let $I_1, \ldots I_k$ be two-sided ideals of R such that $I_i + I_j = R$ for any $i \neq j$ (such ideals are called *comaximal*). Prove there is an isomorphism of R–modules
$$\frac{R}{(I_1 \cap I_2 \cap \cdots \cap I_k)} \cong \frac{R}{I_1} \times \frac{R}{I_2} \times \cdots \times \frac{R}{I_k}.$$

   (b) Prove that for pairwise coprime integers, $m_1, m_2, \ldots, m_k$, there is an isomorphism of groups
$$\mathbb{Z}/m_1 m_2 \cdots m_k\mathbb{Z} \quad \cong \quad \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_k\mathbb{Z}.$$

5. Let M be and R–module with submodules A and B. Prove that the map $A \times B \longrightarrow A + B$ is an isomorphism if and only if $A \cap B = \{0\}$.

6. Give an example of a finitely generated R–module M and a submodule that is not finitely generated.

7. Let S be the set of all sequences of integers $(a_1, a_2, a_3, \ldots)$ that are nonzero in only finitely many components (in other words, all functions $\mathbb{N} \to \mathbb{Z}$ with finite support). Verify that S is a ring (without identity) under componentwise addition and multiplication. Is S a finitely generated S–module?

8. A student makes the following claim: "Since $\mathbb{Z}/2\mathbb{Z}$ is a subring of $\mathbb{Z}/4\mathbb{Z}$, we can let $\mathbb{Z}/2\mathbb{Z}$ act by left multiplication to give $\mathbb{Z}/4\mathbb{Z}$ the structure of a $\mathbb{Z}/2\mathbb{Z}$–module. Then $\mathbb{Z}/4\mathbb{Z}$ is a $\mathbb{Z}/2\mathbb{Z}$–vector space with 4 elements, so it must be isomorphic as a vector space to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$." Prove that $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ are not even isomorphic as abelian groups, and find the flaw in this argument.

9. A *central idempotent* e in a ring R is an central element satisfying $e^2 = e$.

   (a) What are the central idempotents in $\mathbb{Z}^n$?

   (b) What are the central idempotents in $\mathrm{M}_2(\mathbb{Q})$, the 2×2 rational matrices?

   (c) Show that if e is a central idempotent in R and M an R–module, then $M \cong eM \oplus (1 - e)M$.

10. Suppose that R is a ring and that S is a subring.

    (a) Suppose that F is a free R–module. Prove or disprove: F is a free S–module after restriction of scalars to S.

    (b) Suppose that M is an R–module that is free as an S–module after restriction to S. Prove or disprove: M must be a free R–module.

11. Let R be a ring.

    (a) Give the definition of a *free* R–module on a set A.

    (b) Given a set A, explain how to construct a free R–module $F(A)$ on A.

    (c) State the universal property for a free R–module.

    (d) Verify that $F(A)$ satisfies this universal property.

    (e) Prove that the universal property determines $F(A)$ uniquely up to unique isomorphism.

    (f) Show that F defines a covariant functor from the category of sets to the category of R–modules.

12. Let R be a commutative ring, and let A, B, M be R–modules. Use the universal property of the direct sum to prove the isomorphisms of R–modules:
    (a) $\mathrm{Hom}_R(A \oplus B, M) \cong \mathrm{Hom}_R(A, M) \oplus \mathrm{Hom}_R(B, M)$
    (b) $\mathrm{Hom}_R(M, A \oplus B) \cong \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(M, B)$

13. Let R be a commutative ring. If M and N are free R–modules, will the R–module $\mathrm{Hom}_R(M, N)$ be free? If $\mathrm{Hom}_R(M, N)$ is a free R–module, must M and N be free?

14. Find two non-equivalent extensions of the abelian groups $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/6\mathbb{Z}$.

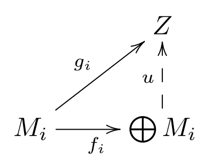
15. Prove that every short exact sequence of vector spaces splits.

16. State the definition of a category, and the definition of a covariant functor.

17. Let $\mathcal{C}$ be a category. Prove that if $X \in \mathrm{ob}(\mathcal{C})$, then the identity morphism id_X is unique. Further prove that if $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, is an isomorphism, then its inverse f^{-1} is unique.

18. (a) Prove that in the category of sets, a map is monic iff it is injective, and epic iff it is surjective.
    (b) Prove that in any category the composition of monomorphisms (respectively, epimorphisms, or isomorphisms) is a monomorphisms (respectively, an epimorphism, or isomorphism).
    (c) Prove that isomorphisms are both monic and epic.

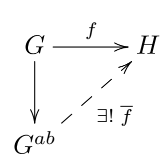
19. **(Coproducts of families).** Prove that the direct sum of R–modules $\bigoplus_{i \in I} M_i$, along with the inclusions $f_i : M_i \to \bigoplus_{i \in I} M_i$, satisfies the following universal property: whenever there is a family of maps $\{g_i : M_i \to Z \mid i \in I\}$ there is a unique map u making the following diagrams commute:



Explain why this universal property can be taken as the definition of the direct sum of R–modules.

20. Let R be a ring. Define a functor on the category R–Mod that takes an R–module M to the R–module $M \oplus M$. Verify that your construction is functorial.

21. **(Abelianization).** Let Grp denote the category of groups and group homomorphisms, and let Ab denote the category of abelian groups and group homomorphisms. Define the *abelianization* G^{ab} of a group G to be the quotient of G by its *commutator subgroup* $[G, G]$, the subgroup normally generated by *commutators*, elements of the form $ghg^{-1}h^{-1}$ for all $g, h \in G$.
    (a) Define a map of categories $[-,-] : \mathrm{Grp} \to \mathrm{Grp}$ that takes a group G to its commutator subgroup $[G, G]$, and a group morphism $f : G \to H$ to its restriction to $[G, G]$. Check that this map is well defined (ie, check that $f([G, G]) \subseteq [H, H]$) and verify that $[-,-]$ is a functor.
    (b) Show that G^{ab} is an abelian group.
    (c) Show that if G is abelian, then $G = G^{ab}$.
    (d) Show that the quotient map $G \to G^{ab}$ satisfies the following universal property: Given any **abelian** group H and group homomorphism $f : G \to H$, there is a unique group homomorphism $\overline{f} : G^{ab} \to H$ that makes the following diagram commute:



Hint: Show that any commutator must be in the kernel of the map f.

This universal property shows that G^{ab} is in a sense the largest abelian quotient of G.

(e) Show that the map ab that takes a group G to its abelianzation G^{ab} can be made into a functor $ab : \underline{\text{Grp}} \to \underline{\text{Ab}}$ by explaining where it maps morphisms of groups $f : G \to H$, and verifying that it is functorial.

22. Let R be a ring. Define the *forgetful functor* from R–modules to abelian groups, and show that it is an exact functor.

23. Recall that a R–module I is called *injective* if the (contravariant) functor $\text{Hom}_R(-, I)$ is exact. Prove that $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$–module, but is not an injective $\mathbb{Z}$–module.

24. The rows of the following diagram are exact. Prove that if m and p are surjective and q is a injective, then n is surjective.

$$\begin{array}{ccccccc} B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B' & \xrightarrow[s]{} & C' & \xrightarrow[t]{} & D' & \xrightarrow[u]{} & E' \end{array}$$

25. Let M be a right R–module, and N a left R–module.
    (a) Describe an explicit construction of the tensor product $M \otimes_R N$ as a quotient of abelian groups.
    (b) State the universal property of the tensor product.
    (c) Verify that the explicit construction satisfies the universal property.

26. Use the universal property of the tensor product $\mathbb{Z}/12\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/20\mathbb{Z}$ to verify that the element $3 \otimes 6$ is nonzero.

27. Define *extension of scalars* to a ring R from a subring S. Show by example that an S–module M may embed into the R–module obtained by extension of scalars, and it may not embed.

28. (a) Suppose that S is a subring of R. Prove that if F is a free S–module on basis A, then $R \otimes_S F$ is a free R–module on basis $\{1 \otimes a \mid a \in A\} \cong A$.
    (b) Conclude that if V is an n-dimensional real vector space on basis $e_1, \ldots, e_n$, then $\mathbb{C} \otimes_{\mathbb{R}} V$ is an n-dimensional complex vector space with basis $1 \otimes e_1, \ldots, 1 \otimes e_n$.

29. What is the complex dimension of the vector spaces $\mathbb{C} \otimes_{\mathbb{R}} \otimes \mathbb{R}^s \otimes_{\mathbb{R}} \mathbb{R}^t$ and $\mathbb{C}^t \otimes_{\mathbb{R}} \mathbb{R}^s$?

30. Prove that any element of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^3$ can be written as the sum of at most two simple tensors (Recall: a *simple* or *pure* tensor in $V \otimes_R W$ is an element of form $v \otimes w$).

31. Let V be a $\mathbb{C}[x]$–module where x acts by a linear transformation A, and let W be a $\mathbb{C}[x]$–module where x acts by a linear transformation B. If V and W have positive dimensions m and n over $\mathbb{C}$, is it possible that $V \otimes_{\mathbb{C}[x]} W$ could be zero? Is it possible that it could be mn-dimensional? Under what conditions could it be less than nm–dimensional?

32. Compute $(\mathbb{Z}/15\mathbb{Z} \oplus \mathbb{R}) \otimes_{\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Q}) \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$.

33. Prove or disprove: Suppose S is a subring of the commutative ring R, and M and N are R–modules. Then the tensor product $M \otimes_R N$ is a quotient of the tensor product $M \otimes_S N$.

34. Let R be an integral domain. Prove or disprove: The map of R–modules that takes an R–module M to its R–submodule $\text{Tor}(M)$ and takes a map $f : M \to N$ to its restriction $f|_{\text{Tor}(M)}$ defines an exact covariant functor R–$\underline{\text{Mod}} \to R$–$\underline{\text{Mod}}$.

35. Let V be a $\mathbb{C}[x]$–module such that V is finite dimensional as a vector space over $\mathbb{C}$. Prove that V is a torsion module.

36. Let R be an integral domain and M an R–module. Suppose that $x_1, \ldots, x_n$ is a maximal list of linearly independent elements. Prove that $Rx_1 + Rx_2 + \cdots + Rx_n$ is isomorphic to R^n, and that $M/(Rx_1 + Rx_2 + \cdots + Rx_n)$ is a torision R–module.

37. Let R be an integral domain. Suppose that A and B are R–modules of ranks a and b, respectively. Prove that $A \oplus B$ is an R–module of rank $a + b$.

38. Let R be an integral domain. Give examples of two non-isomorphic finitely generated torsion R–modules with the same annihilators.

39. Let R be an integral domain, and I any **non-principal** ideal of R. Determine the rank of I, and prove that I is not a free R–module.

40. Find the lists of invariant factors and of elementary divisors for the finitely generated abelian group

$$M \cong \mathbb{Z}^7 \oplus \frac{\mathbb{Z}}{20\mathbb{Z}} \oplus \frac{\mathbb{Z}}{18\mathbb{Z}} \oplus \frac{\mathbb{Z}}{75\mathbb{Z}}.$$