Reading: Dummit–Foote 18.3, Fulton–Harris "Representation Theory: A first course", Ch 2.

Summary of definitions and main results

Definitions we've covered: induced representations

Main results: # complex G–irreps = # G-conjugacy classes, orthogonality of characters, how to use character theory to compute decompositions of characters into irreducibles, Frobenius reciprocity

Warm-Up Questions

1. (a) Compute the character table of the cyclic group $G = \mathbb{Z}/4\mathbb{Z}$,
   (b) Verify the orthogonality relations on the row and columns of the character table.
   (c) Compute the character of $\bigwedge^3 \mathbb{C}[G]$, and determine its decomposition into irreducible characters.
2. Consider the complex S_4–representation $\mathbb{C}^4 \cong \underline{\text{Trv}} \oplus \underline{\text{Std}}$.
   (a) Prove that $\underline{\text{Std}}$ is irreducible.
   (b) Compute the character table of S_4.
   (c) Compute the characters of $\wedge^3\mathbb{C}^4$ and $\text{Sym}^2\underline{\text{Std}}$, and decompose each into irreducible characters.
3. Let G be a finite group.
   (a) Prove that the dimension of the space of class functions $G \to \mathbb{F}$ over $\mathbb{F}$ is equal to the number of conjugacy classes of G.
   (b) Prove that a complex-valued class function on G is a character if and only if it is a nonnegative integer linear combination of irreducible characters.
4. (a) State the two versions of the orthogonality results for the complex character table of a finite group G (the version for rows, and the version for columns).
   (b) Describe the $\mathbb{C}$–vector space of class functions on G, and explain why this space is an inner product space. Explain how this inner product structure relates to the our orthogonality theorem of characters.
   (c) Explain the utility of the orthogonality relations for decomposing G–representations into their irreducible components.
5. Let G be a finite group and C be its character table (of all irreducible characters).
   (a) Show that the "orthogonality of characters" result is equivalent to the statement that the matrix C satisfies the relation $\overline{C}DC^T = Id$ for a certain diagonal matrix D. What is D?
   (b) Conclude from this equation that $C^T\overline{C} = D^{-1}$. Use this equation to derive the second orthogonality result for characters.
   (c) Explicitly verify the relations $\overline{C}DC^T = Id$ and $C^T\overline{C} = D^{-1}$ for the character table for S_3.
6. Prove that the character table is an invertible matrix.
7. Let G be a group, and V and U be irreducible complex representations of G.
   (a) Show by example that $U \otimes_{\mathbb{C}} V$ may or may not be an irreducible G–representation.
   (b) Prove that if U is 1-dimensional, then $U \otimes_{\mathbb{C}} V$ is an irreducible G–representation.

Assignment Questions

1. **(Bonus)** Let G be a finite group. In this question we will describe the ring structure on the group ring $\mathbb{C}[G]$. Let $V_1, \ldots, V_k$ denote a complete list of nonisomorphic irreducible complex G–representations.
   (a) The action of G on a representation V is equivalent to the data of a map of rings $\mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}(V)$, so we obtain a map of rings $\mathbb{C}[G] \to \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{C}}(V_i)$. Show that this map is injective. *Hint:* First show that the regular representation is faithful.
   (b) Conclude (by a dimension count) that there is an isomorphism of rings $\mathbb{C}[G] \cong \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{C}}(V_i)$

2. (a) **(Bonus)** Compute the character table for the symmetric group S_5 over $\mathbb{C}$.
   (b) **(Bonus)** Let $\underline{\mathrm{Std}}$ denote the standard representation of S_5. Use the character table to find the decomposition of $\wedge^3\underline{\mathrm{Std}}$ into irreducible S_5–representations.

3. (**Induced representations**) Suppose $H \subseteq G$ are finite groups, and k is a field. Given a finite dimensional G–representation W, we can restrict the action of G to the action of $H \subset G$. The resulting H–representation is denoted $\mathrm{Res}_H^G W$.

   Conversely, given a finite dimensional group representation V of H over k (viewed as a $k[H]$–module), we can construct a representation of G by extension of scalars. Since $k[H]$ is a subring of $k[G]$, we may view $k[G]$ as a right $k[H]$–module. Define a $k[G]$–module, called the *induced representation* $\mathrm{Ind}_H^G V$, by

   $$\mathrm{Ind}_H^G V := k[G] \otimes_{k[H]} V.$$

   (a) Cite properties of the tensor product to show that

   $$\mathrm{Ind}_H^G(U \oplus U') \cong \mathrm{Ind}_H^G U \oplus \mathrm{Ind}_H^G U' \qquad \text{and} \qquad \mathrm{Ind}_K^G(\mathrm{Ind}_H^K V) \cong \mathrm{Ind}_H^G V$$

   for any representations U, U' of H or subgroups $H \subseteq K \subseteq G$.

   (b) Let G/H be the set of left cosets of G in H, and let $\{\sigma_i\}$ be a set of representatives of each coset. This means for each $g \in G$ and $\sigma_i \in G/H$, there is some $h \in H$ and $\sigma_j \in G/H$ such that $g\sigma_i = \sigma_j h$. Show that $\mathrm{Ind}_H^G V = k[G] \otimes_{k[H]} V$ is isomorphic to the G–representation

   $$\bigoplus_{\sigma_i \in G/H} \sigma_i V$$

   where $\sigma_i V := \{\sigma_i v \mid v \in V\}$ has an action of G by $g(\sigma_i v) = \sigma_j h(v)$.

   (c) Given an G–representation W and H–representation V, find the degrees of $\mathrm{Res}_H^G W$ and $\mathrm{Ind}_H^G V$.

   (d) What representation is $\mathrm{Ind}_H^G V$ when H is the trivial group and $V \cong k$ the trivial representation?

   (e) Let $G = S_n$, $H = S_{n-1}$, and V be the degree 1 trivial S_{n-1}–representation. What is $\mathrm{Ind}_{S_{n-1}}^{S_n} V$?

   (f) **(Ind-Res adjunction)** Prove that induction satisfies the following universal property: If U is any representation of G, then any map of $k[H]$–modules $\phi : V \to \mathrm{Res}_H^G U$ can be promoted uniquely to a map of $k[G]$–modules $\Phi : \mathrm{Ind}_H^G V \to U$, such that Φ restricts to the map ϕ on the subrepresentation

   $$V \cong (id)V \subseteq \bigoplus_{\sigma_i \in G/H} \sigma_i V \cong \mathrm{Ind}_H^G V.$$

   Moreover, every $k[G]$–module map $\mathrm{Ind}_H^G V \to U$ arises in this way. In other words, there is a natural identification of k–modules

   $$\mathrm{Hom}_{k[H]}(V, \mathrm{Res}_H^G U) \cong \mathrm{Hom}_{k[G]}(\mathrm{Ind}_H^G V, U).$$

   Hint: It suffices to show this is a special case of the tensor–hom adjunction from HWK #6 Q4.

   (g) **(Frobenius Reciprocity)** Conclude that for finite dimensional representations over $\mathbb{C}$,

   $$\langle \chi_{\mathrm{Res}_H^G U}, \chi_V \rangle_H = \langle \chi_U, \chi_{\mathrm{Ind}_H^G V} \rangle_G.$$

   Show in particular that if V and U are irreducible representations of H and G, respectively, then the multiplicity of the $k[H]$–representation V in $\mathrm{Res}_H^G U$ is equal to the multiplicity of the $k[G]$–representation U in $\mathrm{Ind}_H^G V$.