Reading: Dummit–Foote 18.1 (up to page 846) & 12.1 (Definition and Theorem 1) & Ch 10.3.

Summary of definitions and main results

Definitions we've covered: group ring, (linear) representation, degree of a representation, faithful representation, trivial representation, permutation representation, regular representation, homomorphism and isomorphism of representations, G-equivariant map, intertwiner, generators of an R–module, the R–submodule RA generated by a set A, finite generation, cyclic module, ascending chain condition (ACC), Noetherian R–module, Noetherian ring, minimal set of generators, direct product, direct sum (externel and internal), free module, basis, rank of a free module

Main results: Equivalent definitions of a group representation, examples of non-Noetherian modules, equivalent definitions of Noetherian module, equivalent definitions of (internal) direct sums

Warm-Up Questions

The "warm-up" questions do not need to be submitted (and won't be graded).

1. Let G be a group and V an $\mathbb{F}$-vector space. Show that the following are all equivalent ways to define a (linear) representation of G on V.
   i. A group homomorphism $G \to \mathrm{GL}(V)$.
   ii. A group action (by linear maps) of G on V.
   iii. An $\mathbb{F}[G]$–module structure on V.
2. Let R be a commutative ring. Show that the group ring $R[\mathbb{Z}] \cong R[t, t^{-1}]$. Show that $R[\mathbb{Z}/n\mathbb{Z}] \cong R[t]/\langle t^n - 1\rangle$.What is the group ring $R[\mathbb{Z}^n]$? The group ring $R[\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}]$?
3. Let $\phi : G \to GL(V)$ be any group representation. What is the image of the identity element in $GL(V)$?
4. Compute the sum and product of $(1+3e_{(1\,2)}+4e_{(1\,2\,3)})$ and $(4+2e_{(1\,2)}+4e_{(1\,3)})$ in the group ring $\mathbb{Q}[S_3]$.
5. Let G be a group and R a commutative ring. Show that $R[G]$ is commutative if and only if G is abelian.
6. Given any representation $\phi : G \to GL(V)$, prove that ϕ defines a faithful representation of $G/\ker(\phi)$.
7. (a) Find an explicit isomorphism T between the following two representations of S_2.

$$S_2 \to GL(\mathbb{R}^2) \qquad (1\ 2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad S_2 \to GL(\mathbb{R}^2) \qquad (1\ 2) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   Give a geometric description of the action and the bases for $\mathbb{R}^2$ associated to each matrix group.

   (b) Prove that the following two representations of S_2 are not isomorphic.

$$S_2 \to GL(\mathbb{R}^2) \qquad (1\ 2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad S_2 \to GL(\mathbb{R}^2) \qquad (1\ 2) \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

8. Let A and B be submodules of the R–module M. Show that $A + B$ is equal to $R(A \cup B)$, the submodule generated by $A \cup B$, as R–submodules of M.
9. Let R be a ring and I a two-sided ideal of R. For each of the following R–modules M indicate whether M is finitely generated, cyclically generated, or more information is needed:
   $M = R^n$ for $n \in \mathbb{N}$, polynomials $M = R[x]$, series $M = R[[x]]$, $M = I$, and $M = R/I$.

10. (a) Prove that if M is a finitely generated R–module, and $\phi : M \to N$ a map of R–modules, then its image $\phi(M)$ is finitely generated by the images of the generators. Conclude in particular that all quotients of finitely generated modules are finitely generated.
    (b) Let M be an R–module and N a submodule. Prove that if both N and M/N are finitely generated R–modules, then M is a finitely generated R–module.

11. (a) Let $\mathbb{F}$ be a field. Citing results from linear algebra, explain why every $\mathbb{F}$–module is Noetherian.
    (b) Citing results from group theory, explain why $\mathbb{Z}$–module is Noetherian.
    (c) Explain why all PIDs are Noetherian rings.

12. An R–submodule N of an R-module M has a *direct complement* P if $M \cong N \oplus P$.
    (a) Show that the $\mathbb{Z}$–submodule $2\mathbb{Z} \subseteq \mathbb{Z}$ does not have a direct complement.
    (b) Let V be the the $\mathbb{Q}[x]$–module where x acts by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Show that $U = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ is a submodule of V with no direct complement.
    (c) Show that every linear subspace of a vector space has a direct complement.

13. (a) Let A be any finite set of n elements. Show that the free R–module on A is isomorphic as an R–module to R^n.
    (b) For R commutative, are the polynomial rings $R[x]$ and $R[x, y]$ free R–modules? What about Laurent polynomials $R[x, x^{-1}]$? Rational functions in x?
    (c) Do these arguments work for series $R[[x]]$?

14. Show that $M = \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ is a free $\mathbb{Z}/10\mathbb{Z}$–module by finding a basis. Show that the element $(2, 2)$ cannot be an element of any basis for M. Is the submodule $N = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ also free?

15. Let $\{M_i \mid i \in I\}$ be a (possibly infinite) set of R–modules. Prove that the direct sum $\bigoplus_{i \in I} M_i$ is a submodule of the direct product $\prod_{i \in I} M_i$, but show by example that these may not be isomorphic in general. *Hint*: What are their cardinalities?

16. Fix an integer $n > 0$. Recall the following example from class: The symmetric group S_n acts on $\mathbb{C}^n$ by permuting a basis $e_1, e_2, \ldots, e_n$. We saw that this representation has two subrepresentations,
    $$D = \text{span}_{\mathbb{C}}(e_1 + e_2 + \cdots e_n) \qquad \text{and} \qquad U = \{a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \mid a_1 + a_2 + \cdots a_n = 0\}.$$
    (a) Show that U and D are *simple*, that is, they do not contain any nontrivial subrepresentations.
    (b) Show that, as a $\mathbb{C}S_n$–module, $\mathbb{C}^n$ is the direct sum $\mathbb{C}^n \cong D \oplus U$.

    Later in the course we will prove the following incredible fact: Finite dimensional representations of finite groups over $\mathbb{C}$ *always* decompose into a direct sum of simple subrepresentations.

17. **(Group theory review)** For which $m, n \in \mathbb{Z}$ will the group $(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})$ be cyclically generated?

18. **(Linear algebra review)** Let V, W be vector spaces over a field $\mathbb{F}$ of dimension n and m, respectively.
    (a) Consider a linear map $A : V \to V$ (equivalently, of an $n \times n$ matrix A). Show that the following are equivalent. If A satisfies any of these conditions, it is called *singular*.
        1. A has a nontrivial kernel
        2. $\text{rank}(A) < n$
        3. A is not invertible
        4. The columns of A are linearly dependent
        5. The rows of A are linearly dependent
        6. $\det(A) = 0$
        7. $\lambda = 0$ is an eigenvalue of A
    (b) Let T be a linear transformation on a finite-dimensional $\mathbb{F}$-vector space V. Show that the following are equivalent
        1. λ is an eigenvalue of T
        2. $(\lambda I - T)$ is singular
        3. λ is a root of the *characteristic polynomial* of T, $p_T(x) = \det(xI - T)$.

Assignment Questions

1. Let G be a finite group, and $\mathbb{F}$ a field.
   The following computations will be significant when we study *character theory* later in Math 122.
   (a) Let $G \to GL(U)$ be any representation of G. Citing facts from linear algebra (which you don't need to prove), explain why the trace of the matrix representing a given element $g \in G$ is well-defined in the sense that it will be the same in any isomorphic representation of G.
   (b) A *permutation representation* of G on a finite-dimensional $\mathbb{F}$-vector space V is a linear representation $\rho : G \to GL(V)$ in which elements act by permuting some basis $B = \{b_1, \ldots b_m\}$ for V. Show that, with respect to the basis $\{b_1, \ldots, b_m\}$, for each element $g \in G$, $\rho(g)$ is represented by an $m \times m$ *permutation matrix*, a square matrix that has exactly one entry 1 in each row and each column, and zero elsewhere. Use this description of matrices $\rho(g)$ to show that the trace of $\rho(g)$ is equal to the number of basis elements b_i fixed by $\rho(g)$.
   (c) The group ring of $\mathbb{F}[G]$ is a left module over itself. Show that this corresponds to permutation representation of the group G on the underlying vector space $\mathbb{F}[G]$, called the *(left) regular representation* of G. Find the degree of this representation. In what basis is this a permutation representation, and how many G-orbits does this basis have?
   (d) For any $g \in G$, compute the trace of the matrix representing g in the regular representation.

2. Suppose a finitely generated R–module M has a minimal generating set $A = \{a_1, a_2, \ldots, a_n\}$.
   Prove or find a counterexample: $M \cong Ra_1 \oplus Ra_2 \oplus \cdots \oplus Ra_n$.

3. (a) **(Chinese Remainder Theorem)** Let R be any ring, and let $I_1, \ldots I_k$ be two-sided ideals of R such that $I_i + I_j = R$ for any $i \neq j$ (such ideals are called *comaximal*). Prove there is an isomorphism of R–modules
   $$\frac{R}{(I_1 \cap I_2 \cap \cdots \cap I_k)} \cong \frac{R}{I_1} \times \frac{R}{I_2} \times \cdots \times \frac{R}{I_k}.$$
   (b) Prove that for pairwise coprime integers, $m_1, m_2, \ldots, m_k$, there is an isomorphism of groups
   $$\mathbb{Z}/m_1 m_2 \cdots m_k \mathbb{Z} \quad \cong \quad \mathbb{Z}/m_1\mathbb{Z} \times \mathbb{Z}/m_2\mathbb{Z} \times \cdots \times \mathbb{Z}/m_k\mathbb{Z}.$$

4. Let R be a ring. Show that an arbitrary direct sum of free R–modules is free, but an arbitrary direct product need not be. *Hint:* Dummit–Foote 10.3 # 24.

5. (a) Let R be a commutative ring, and let $n, m \in \mathbb{N}$. Prove that that $R^n \cong R^m$ if and only if $n = m$. You may assume without proof that finite-dimensional vector spaces are isomorphic if and only if their dimensions are equal. You may also assume Zorn's Lemma.
   Hint: See Dummit–Foote 10.3 # 2.
   (b) Show that when R is not commutative, this statement is false – that is, free R–modules need not have a unique rank. *Hint:* See Dummit–Foote 10.3 # 27.

6. **(Bonus)** Let V be an $\mathbb{C}[x]$–module with V finite dimensional over $\mathbb{C}$, and x acting by the linear map T. For which linear maps T will V be cyclically generated? Give necessary and sufficient conditions on the eigenvalues and eigenspaces of T. Remember that not all linear maps are diagonalizable!