Reading: Dummit–Foote Ch. 10.3 & pp 911-913.

Summary of definitions and main results

Definitions we've covered: R–linear independence, category, object, morphism, monomorphism, epimorphism, isomorphism, universal property, covariant and contravariant functors, forgetful functor, free functor.

Main results: Universal property for free modules, construction of the free module $F(A)$, verification that $F(A)$ satisfies the universal property, universal properties define objects up to unique isomorphism, in the category R-mod monomorphisms are precisely the injections, free functor $F : \underline{\text{Set}} \to R\text{–}\underline{\text{Mod}}$ is functorial.

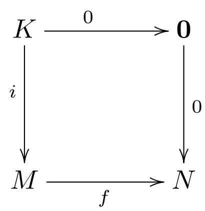
Warm-Up Questions

The "warm-up" questions do not need to be submitted (and won't be graded).

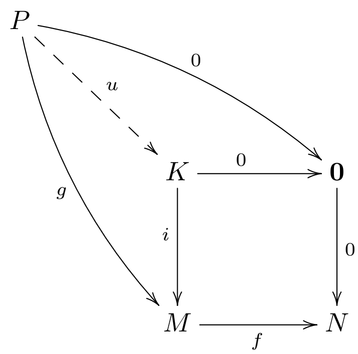
1. Show that $M = \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$ is a rank-2 free module over $\mathbb{Z}/6\mathbb{Z}$, and find necessary and sufficient conditions on a pair of elements $\{a, b\} \subset M$ to be a basis for M.
2. Let R be a ring, M and R–submodule, and $A \subseteq M$ a set. The set A is called *R–linearly independent* if any finite R–linear combination $\sum_i r_i a_i$ (with $a_i \in A, r_i \in R$) is equal to zero if and only if every coefficient r_i is zero.
   (a) Show that A is a basis for the R–module RA it generates if and only if A is R–linearly independent.
   (b) Find a counterexample to the following false statement: If M is a free R–module and $A \subseteq M$ is an R–linearly independent subset of M, then A can be extended to a basis for M.
3. Let F be the free R–module on a set A. Show that if R has no zero divisors and $N \subseteq M$ is any nonzero submodule, then $\text{ann}(N) = \{0\}$. Is this true when R has zero divisors?
4. In class (and in Dummit-Foote 10.3 Theorem 6) we gave a construction of a free module $F(A)$ on a set A. Verify that this construction is in fact a free module with basis A (as given in the definition on p354). Show moreover that $F(A) \cong \bigoplus_A R$.
5. (a) Citing results from linear algebra, explain why every vector space over a field $\mathbb{F}$ is a free $\mathbb{F}$–module.
   (b) When $\mathbb{F}$ is a field, any minimal finite generating set $B = \{a_1, \ldots, a_n\}$ of an $\mathbb{F}$–module must be linearly independent and therefore a basis. Prove that in general, if an R–module has a minimal generating set $B = \{a_1, \ldots, a_n\}$, then R need not be free on B.
   (c) Suppose that M is an R–module containing elements $\{a_1, a_2, \ldots, a_n\}$ such that $M = Ra_1 \oplus Ra_2 \oplus \cdots \oplus Ra_n$. Explain how $A = \{a_1, a_2, \ldots a_n\}$ could fail to be a basis for M. What conditions on the elements a_i could ensure that A is a basis?
6. Let R be a ring, M and R–module and N an R–submodule of N.
   (a) Show that M/N satisfies the following universal property: If $\varphi : M \to Q$ is any map of R–modules satisfying $\phi(n) = 0$ for all $n \in N$, then φ factors uniquely through M/N.
   (b) Show that this universal property defines the quotient M/N uniquely up to unique isomorphism.
7. (a) Prove that in the category of R–modules, a morphism is epic if and only if it is a surjective map.
   (b) Prove that in the category of rings, the map $\mathbb{Z} \to \mathbb{Q}$ is an epic morphism that is not surjective.
8. (a) A *zero object* $\mathbf{0}$ in a category is an object with the following property: For any object M, there is a unique morphism from M to $\mathbf{0}$, and a unique morphism from $\mathbf{0}$ to M. Show that if a category has a zero object, then it is unique up to unique isomorphism.

(b) Let $\mathcal{C}$ be the category of R–modules, and show that the zero module $\{0\}$ is a zero object. This definition allows us to define the *zero map* 0 between R–modules M and N: it is the composition of the unique map $M \to \mathbf{0}$ with the unique map $\mathbf{0} \to N$.

(c) Let $\mathcal{C}$ be the category R–modules. Verify that the kernel of an R–module map satisfies the following universal property. If $f : M \to N$ is a morphism in $\mathcal{C}$, then define the *kernel* $i : K \to M$ of f to be the map i such that $f \circ i$ is the zero morphism 0



and satisfying the following: whenever there is a map of R–modules $g : P \to M$ such that $f \circ g = 0$, there is a unique map $u : P \to K$ such that $i \circ u = g$. In other words, there is a unique map u that makes the following diagram commute.



(d) Explain why this univeral property determines the map $i : K \to M$ up to unique isomorphism. Conclude that this universal property can be taken as the definition of the kernel of f.

9. Let $\mathscr{C}$ be a category containing objects A and B, and let F be a functor $F : \mathscr{C} \to \mathscr{D}$. Show that if A and B are isomorphic objects of $\mathscr{C}$, then $F(A)$ and $F(B)$ will be isomorphic objects of $\mathscr{D}$.

10. Given a group G, define a category $\mathscr{G}$ with a single object $\bigstar$ and morphisms $\mathrm{Hom}_{\mathscr{G}}(\bigstar, \bigstar) = \{g \mid g \in G\}$. The composition law is given by the group operation. Show that a function between groups $G \to H$ is a group homomorphism if and only if the corresponding map between categories $\mathscr{G} \to \mathscr{H}$ is a functor.

11. Let $\underline{\text{f Set}}$ denote the category of finite sets and all functions between sets. Let $\mathscr{P} : \underline{\text{f Set}} \to \underline{\text{f Set}}$ be the function that takes a finite set A to its *power set* $\mathscr{P}(A)$, the set of all subsets of A. If $f : A \to B$ is a function of finite sets, let $\mathscr{P}(f) : \mathscr{P}(A) \to \mathscr{P}(B)$ be the function that takes a subset $U \subseteq A$ to the subset $f(U) \subseteq B$.
   (a) Show that $\mathscr{P}$ is a covariant functor.
   (b) What if we had instead defined $\mathscr{P}(f) : \mathscr{P}(B) \to \mathscr{P}(A)$ to take a subset $U \subseteq B$ to its preimage $f^{-1}(U) \subseteq A$ under f?

12. Let 0 denote the trivial abelian group. Give an example of a functor $F : \underline{\text{Ab}} \to \underline{\text{Ab}}$ such that $F(0) = 0$, and a functor $F : \underline{\text{Ab}} \to \underline{\text{Ab}}$ such that $F(0) \neq 0$.

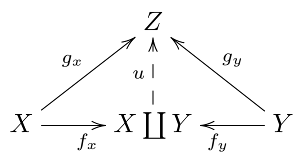
13. Let $\underline{\text{Grp}}$ be the category of groups and group homomorphisms. Let Z be the map $Z : \underline{\text{Grp}} \to \underline{\text{Grp}}$ that maps a group G to its centre $Z(G) = \{a \in G \mid ag = ga \ \ \forall g \in G\}$. Show that Z **cannot** be made into a functor by defining it to take a map of groups $f : G \to H$ to the restriction $f|_{Z(G)}$ of f to $Z(G)$, since $f(Z(G))$ may not be contained in $Z(H)$.

Assignment Questions

1. Let N and M_i be R–modules for i in an index set I. Prove the following isomorphisms of abelian groups:

   (a) $\mathrm{Hom}_R\left(N, \prod_{i\in I} M_i\right) \cong \prod_{i \in I} \mathrm{Hom}_R(N, M_i)$ (b) $\mathrm{Hom}_R\left(\bigoplus_{i\in I} M_i,\ N\right) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i, N)$

2. **(Coproducts).** Let $\mathcal{C}$ be a category with objects X and Y. The *coproduct* of X and Y (if it exists) is an object $X \coprod Y$ in $\mathcal{C}$ with maps $f_x : X \to X \coprod Y$ and $f_y : Y \to X \coprod Y$ satisfying the following universal property: whenever there is an object Z with maps $g_x : X \to Z$ and $g_y : Y \to Z$, there exists a unique map $u : X \coprod Y \to Z$ that makes the following diagram commute:



   (a) Let X and Y be objects in $\mathcal{C}$. Show that, if the coproduct $(X \coprod Y, f_x, f_y)$ exists in $\mathcal{C}$, then the universal property determines it uniquely up to unique isomorphism.

   (b) Prove that in the category of R–modules, the coproduct of R–modules $X \coprod Y$ is $X \oplus Y$ with the canonical inclusions of X and Y. In other words, this universal property defines the direct sum operation on R–modules.

   (c) Prove that in the category of groups, the univeral property for the coproduct $X \coprod Y$ of groups X and Y does *not* define the direct product of those groups. (It is a construction called the *free product* of groups).

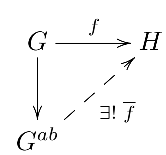
   (d) Prove that in the category of sets, the coproduct $X \coprod Y$ of sets X and Y is their disjoint union.

3. **(Abelianization).** Let Grp denote the category of groups and group homomorphisms, and let Ab denote the category of abelian groups and group homomorphisms. Define the *abelianization* G^{ab} of a group G to be the quotient of G by its *commutator subgroup* $[G, G]$, the subgroup normally generated by *commutators*, elements of the form $ghg^{-1}h^{-1}$ for all $g, h \in G$.

   (a) Define a map of categories $[-, -] : \underline{\mathrm{Grp}} \to \underline{\mathrm{Grp}}$ that takes a group G to its commutator subgroup $[G, G]$, and a group morphism $f : G \to H$ to its restriction to $[G, G]$. Check that this map is well defined (ie, check that $f([G, G]) \subseteq [H, H]$) and verify that $[-, -]$ is a functor.

   (b) Show that G^{ab} is an abelian group. Show moreover that if G is abelian, then $G = G^{ab}$.

   (c) Show that the quotient map $G \to G^{ab}$ satisfies the following universal property: Given any **abelian** group H and group homomorphism $f : G \to H$, there is a unique group homomorphism $\overline{f} : G^{ab} \to H$ that makes the following diagram commute:



   This universal property shows that G^{ab} is in a sense the "largest" abelian quotient of G.

   (d) Show that the map ab that takes a group G to its abelianization G^{ab} can be made into a functor $ab : \underline{\mathrm{Grp}} \to \underline{\mathrm{Ab}}$ by explaining where it maps morphisms of groups $f : G \to H$, and verifying that it is functorial.

(e) The category $\underline{\text{Ab}}$ is a subcategory of $\underline{\text{Grp}}$. Define the functor $\mathcal{A} : \underline{\text{Ab}} \to \underline{\text{Grp}}$ to be the inclusion of this subcategory; $\mathcal{A}$ takes abelian groups and group homomorphisms in $\underline{\text{Ab}}$ to the same abelian groups and the same group homomorphisms in $\underline{\text{Grp}}$. Briefly explain why the universal property in Part (c) can be rephrased as follows: Given groups $G \in \underline{\text{Grp}}$ and $H \in \underline{\text{Ab}}$, there is a natural bijection between the sets of morphisms:

$$\text{Hom}_{\underline{\text{Grp}}}(G, \mathcal{A}(H)) \cong \text{Hom}_{\underline{\text{Ab}}}(G^{ab}, H)$$

(This means that $\mathcal{A} : \underline{\text{Ab}} \to \underline{\text{Grp}}$ and $ab : \underline{\text{Grp}} \to \underline{\text{Ab}}$ are what we call a pair of *adjoint functors*.)

4. Define a ring R to be *(left) Noetherian* if R is Noetherian as a left module over itself. In this question we will show this definition is equivalent to the following alternate definition of a Noetherian ring: R is *(left) Noetherian* if **every** finitely generated left R–module is Noetherian.

(a) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R–modules. Show that if A and C are finitely generated R–modules, then B is finitely generated.

(b) Suppose R is Noetherian as a left R–module. Let M be a submodule of R^n. Consider the short exact sequence of R–modules

$$0 \longrightarrow \{0\} \times R^{n-1} \longrightarrow R^n \xrightarrow{\pi_1} R \longrightarrow 0,$$

(Here, π_1 is the projection onto the first factor of R^n.) Show that we obtain a short exact sequence

$$0 \longrightarrow M \cap \big(\{0\} \times R^{n-1}\big) \longrightarrow M \longrightarrow \pi_1(M) \longrightarrow 0.$$

(c) Using parts (a) and (b) and induction on n, prove that R^n is a Noetherian R–module.

(d) Prove that an R–module N is finitely generated if and only if it is quotient of a finite rank free R–module R^n.

(e) Prove that a quotient of a Noetherian R–module is Noetherian.

(f) Conclude that any finitely generated R–module is Noetherian.