Reading: Dummit–Foote Ch 10.5 up to p402 (the book goes into more detail on these topics than we will).

Summary of definitions and main results

Definitions we've covered: Exact, exact sequence, short exact sequence, extension of C by A, extension problem, presentation, relations, homomorphism of short exact sequences, isomorphism of short exact sequences, equivalence of extensions, split short exact sequence, exact functor, right exact functor, left exact functor, functors $\mathrm{Hom}_R(D,-)$ and $\mathrm{Hom}_R(-,D)$.

Main results: Short Five Lemma, Splitting Lemma, $\mathrm{Hom}_R(D,-)$ is a covariant functor, $\mathrm{Hom}_R(-,D)$ is a contravariant functor, $\mathrm{Hom}_R(D,-)$ is left exact.

Warm-Up Questions

1. Let R be a ring. Consider the map on the objects of R–Mod that takes and R–module M to the submodule ann(R), and takes a morphism of R–modules $f : M \to N$ to its restriction $f|_{\mathrm{ann}(R)}$ to the submodule $\mathrm{ann}(R) \subseteq M$. Does this give a well-defined functor R–Mod $\to$ R–Mod?
2. Write down short exact sequences giving presentations of the following R–modules M. Give a list of generators and relations for M.
   (a) R^n
   (b) $R = \mathbb{Z}$, $M = \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$
   (c) $R = \mathbb{Q}$, $M = \mathbb{Q}[x]/\langle x^2 + 1\rangle$
   (d) $R = \mathbb{C}[x, y]$, $M = \langle x, y\rangle$
3. (a) Find two nonequivalent extensions of $\mathbb{Z}$–modules $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$.
   (b) Find two nonequivalent extensions of $\mathbb{Z}$–modules $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$.
   (c) How many extensions of $\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$ can you find?
4. (a) Show that if $0 \to U \to W \to V \to 0$ is a short exact sequence of vector spaces, then $W \cong V \oplus U$.
   (b) Show that any two extensions of vector spaces V by U are isomorphic.
5. Use the Splitting Lemma to show that if m and n are coprime, the following short exact sequence splits:

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/mn\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

   What if m and n are not coprime?
6. Show by example that isomorphic extensions need not be equivalent. *Hint:* Page 382, Example (5).
7. (a) Let R be a ring, and let R–Mod be the category of R–modules. Let Ab be the category of abelian groups. Show that there is a covariant functor $\mathscr{F} : R$–Mod $\to$ Ab that maps an R–module M to its underlying abelian group $\mathscr{F}(M)$. (This is an example of a *forgetful functor*, since it forgets the extra data of the action of R on M).
   (b) Explain why this functor is exact.
8. Recall the abelianization functor ab : Grp $\to$ Ab from Assignment #4. Show that ab is right exact but not left exact.

Assignment Questions

1. **(Short Five Lemma).** Consider a homomorphism of short exact sequences of R–modules:

$$\begin{CD}
0 @>>> A @>\psi>> B @>\varphi>> C @>>> 0\\
@. @VV\alpha V @VV\beta V @VV\gamma V @.\\
0 @>>> A' @>>\psi'> B' @>>\varphi'> C' @>>> 0
\end{CD}$$

Prove the remaining step in the Short Five Lemma: If α and γ both surject, then β must also surject.

2. **(The Splitting Lemma).** Let R be a ring, and consider the short exact sequence of R–modules:

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \longrightarrow 0.$$

Prove that the following are equivalent.

(i) The sequence *splits*, that is, B is isomorphic to $A \oplus C$ such that ψ corresponds to the natural inclusion of A, and φ corresponds to the natural projection onto C.

(ii) There is a map $\varphi' : C \to B$ such that $\varphi \circ \varphi'$ is the identity on C.

$$0 \longrightarrow A \xrightarrow{\psi} B \underset{\varphi'}{\overset{\varphi}{\rightleftarrows}} C \to 0$$

(iii) There is a map $\psi' : B \to A$ such that $\psi' \circ \psi$ is the identity on A.

$$0 \to A \underset{\psi'}{\overset{\psi}{\rightleftarrows}} B \xrightarrow{\varphi} C \longrightarrow 0$$

The maps φ' and ψ' are called *splitting homomorphisms.*

3. We proved in class that the map $\text{Hom}_R(D, -) : R\text{–}\underline{\text{Mod}} \to \underline{\text{Ab}}$ is a covariant, left exact functor.

(a) To which groups does the functor $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ map the $\mathbb{Z}$–modules $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $(\mathbb{Z}/n\mathbb{Z})^p$, $\mathbb{Z}/n^p\mathbb{Z}$, and $\mathbb{Z}/m\mathbb{Z}$ (for m, n coprime)? Express your answers in terms of the classification of finitely generated abelian groups.
(No justification needed for Part (a), but you should understand why these isomorphisms hold)

(b) Describe the sequence of abelian groups and the maps obtained by applying $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$ to the following short exact sequences:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

(c) Given any positive integer n, show that the functor $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ is not exact.
Note: An R–module P is called *projective* if the functor $\text{Hom}_R(P, -)$ is an exact functor. This problem proves that $\mathbb{Z}/n\mathbb{Z}$ is **not** a projective $\mathbb{Z}$–module.