Reading: Dummit–Foote Ch 10.4, 10.5, 11.3.

Summary of definitions and main results

Definitions we've covered: Tensor products, R-balanced map, (S, R)–bimodule and R–bimodule, universal property of the tensor product, extension of scalars, tensor product of R-linear maps.

Main results: $\mathrm{Hom}_R(-, D)$ is left exact, explicit construction of $M \otimes_R N$, verification that it satisfies the universal property, $R/I \otimes_R N \cong N/IN$, tensor product distributes over direct sums, $R^n \otimes_R N \cong N^n$, tensor product is associative, hom-tensor adjunction.

Warm-Up Questions

1. Let R be a ring, let A be a right R–module and B a left R–module. Prove that the universal property of the tensor product defines $A \otimes_R B$ uniquely up to unique isomorphism.

2. Explain why, when R is commutative, a left R–module M will also be a right R–module under the action $mr = rm$, and conversely any right R–module N will also have an induced left R–module structure. Will these actions automatically give an R–bimodule structure? Why will these constructions generally not work when R is non-commutative?

3. Let R be a ring with right R–module M and left R–module N. Show that the natural map

$$M \times N \longrightarrow M \otimes_R N$$

is **not** a group homomorphism. What are the constraints on this map, as imposed by the defining relations of $M \otimes_R N$?

4. Let R and S be rings (possibly the same ring). Let M be a right R–module and N a left R–module. When will the tensor product $M \otimes_R N$ have the structure of an abelian group, and under what conditions will it additionally have the structure of an S–module?

5. Let R be a commutative ring. Let e_1, e_2, e_3 be a basis for the R^3 and let f_1, f_2, f_3, f_4 be a basis for R^4. Expand the tensor

$$(a_1e_1 + a_2e_2 + a_3e_3) \otimes (b_1f_1 + b_2f_2 + b_3f_3 + b_4f_4) \qquad \in R^3 \otimes_R R^4.$$

6. Let R be a ring with right R–module M and left R–module N. Which of the following maps are R–balanced? Which are homomorphisms of abelian groups? For the maps that are R–balanced, describe how they factor through the tensor product.
   (a) The identity map $M \times N \longrightarrow M \times N$.
   (b) The natural projections of $M \times N$ onto M and N.
   (c) The natural map $M \times N \longrightarrow M \otimes_R N$.
   (d) Suppose M and N are ideals of R. The multiplication map

$$\begin{aligned} M \times N &\longrightarrow R \\ (m, n) &\longmapsto mn \end{aligned}$$

   (e) Suppose R is commutative. The matrix multiplication map

$$\begin{aligned} M_{n\times k}(R) \times M_{k\times m}(R) &\longrightarrow M_{n\times m}(R) \\ (A, B) &\longmapsto AB \end{aligned}$$

(f) Suppose R is commutative and M, N, P are R–modules. The composition map:

$$\begin{aligned} \mathrm{Hom}_R(M,N) \times \mathrm{Hom}_R(N,P) &\longrightarrow \mathrm{Hom}_R(M,P) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

(g) Suppose R is commutative. The dot product map:

$$\begin{aligned} R^n \times R^n &\longrightarrow R \\ (v,w) &\longmapsto v \cdot w \end{aligned}$$

(h) Suppose R is commutative. The cross product map:

$$\begin{aligned} R^3 \times R^3 &\longrightarrow R^3 \\ (v,w) &\longmapsto v \times w \end{aligned}$$

(i) Suppose R is commutative. The determinant map:

$$\begin{aligned} R^2 \times R^2 &\longrightarrow R \\ (v,w) &\longmapsto \det \begin{bmatrix} | & | \\ v & w \\ | & | \end{bmatrix} \end{aligned}$$

7. Let R be a ring with right R–module M and left R–module N.
   (a) What is the additive identity in $M \otimes_R N$? Show that the simple tensors $0 \otimes n$ and $m \otimes 0$ will be zero in any tensor product $M \otimes_R N$.
   (b) Show there are always maps of abelian groups $N \to M \otimes_R N$, but that these maps may not be injective.

8. Let $V \cong \mathbb{C}^2$ be a complex vector space, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with respect to the standard basis e_1, e_2. Write down the matrix for the linear map induced by A on the four-dimensional vector space $V \otimes V$ with respect to the basis $e_1 \otimes e_1,\ e_1 \otimes e_2,\ e_2 \otimes e_1,\ e_2 \otimes e_2$.

9. Let V be a complex vector space. Let $T : V \to V$ be a diagonalizable linear map with eigenbasis $v_1, v_2, \ldots v_n$, and associated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. What are the eigenvalues of the map induced by T on $V \otimes V$, and what are the associated eigenvectors?

10. Let R be a ring and S a subring.
    (a) Give an example of R, S and an S–module that embeds into a R–module.
    (b) Give an example of R, S, and an S–module that cannot embed into any R–module.

11. (a) Suppose that A is a finite abelian group. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$.
    (b) Suppose that B is a finitely-generated abelian group. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} B$ is a $\mathbb{Q}$–vector space. What determines its dimension?

12. Let M be a right R–module and $N_1, \ldots, N_n$ a set of left R–modules. Verify that the tensor product distributes over direct sums (Dummit–Foote 10.4 Theorem 17). There is a unique group isomorphism

$$M \otimes_R (N_1 \oplus \cdots \oplus N_n) \cong (M \otimes_R N_1) \oplus \cdots \oplus (M \otimes_R N_n).$$

    Conclude that if N is a left R–module, $R^n \otimes_R N \cong N^n$.

13. (a) Let R be a ring, I a left ideal of R, and N a left R–module. Prove that $R/I \otimes_R N \cong N/IN$.

(b) Let R be a commutative ring with ideals I and J. Prove the isomorphism of R–modules:

$$\begin{aligned} R/I \otimes_R R/J &\longrightarrow R/(I+J) \\ (r+I)\otimes(s+J) &\longmapsto rs + (I+J) \end{aligned}$$

14. Verify the associativity of the tensor product (Dummit-Foote 10.4 Theorem 14).

15. **(Linear Algebra Review).** Let V be a finite dimensional vector space over $\mathbb{C}$. Recall that a *Hermitian inner product* on V is a function

$$\langle -, - \rangle : V \times V \to \mathbb{C}$$

satisfying the following properties:

(1) (Conjugate symmetry)

$$\langle x, y\rangle = \overline{\langle y, x\rangle} \qquad \forall x, y \in V$$

(2) (Linearity in the first coordinate)

$$\langle ax, y\rangle = a\langle x, y\rangle \qquad \text{and} \qquad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \qquad \forall x, y, z \in V, a \in \mathbb{C}$$

(3) (Positive definiteness)

$$\langle x, x\rangle \geq 0 \qquad \text{and} \qquad \langle x, x\rangle = 0 \Rightarrow x = 0 \qquad \forall x \in V$$

Observe that (1) and (2) imply that the Hermitian inner product is *antilinear* in the second coordinate:

$$\langle x, ay\rangle = \overline{a}\langle x, y\rangle \qquad \text{and} \qquad \langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle \qquad \forall x, y, z \in V, a \in \mathbb{C}$$

(a) Suppose that there is set of vectors $e_1, e_2, \ldots, e_n$ in V that is *orthonormal* with respect to the inner product $\langle \cdot, \cdot \rangle$. This means

$$\langle e_i, e_j\rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Prove these vectors are linearly independent, and therefore form a basis for the space they span. (NB: We can always use the *Gram-Schmidt algorithm* to construct an orthonormal basis for V.)

(b) Let $v = a_1e_1 + \cdots + a_ne_n$ and $w = b_1e_1 + \cdots + b_ne_n$ be elements of V. Compute $\langle v, w\rangle$. Find in particular the values of $\langle v, e_i\rangle$ and $\langle v, v\rangle$.

(c) Show that the function

$$\begin{aligned} || - || : V &\longrightarrow \mathbb{R}_{\geq 0} \\ ||v|| &= \sqrt{\langle v, v\rangle} \end{aligned}$$

defines a norm on V, and hence the function

$$\begin{aligned} d : V \times V &\longrightarrow \mathbb{R}_{\geq 0} \\ d(v, w) &= ||v - w|| \end{aligned}$$

defines a metric on V.

Assignment Questions

1. **(The functor** $\mathrm{Hom}_R(-,D)$**).**
   (a) Show that if D is any R–module, then there is a **contravariant** functor

$$\begin{aligned}\mathrm{Hom}_R(-,D) : R\text{–}\underline{\mathrm{Mod}} &\longrightarrow \underline{\mathrm{Ab}}\\ M &\longmapsto \mathrm{Hom}_R(M,D)\\ [\phi : M \to N] &\longmapsto \left[\begin{smallmatrix}\phi^* : \mathrm{Hom}_R(N,D) \longrightarrow \mathrm{Hom}_R(M,D)\\ f \longmapsto f\circ\phi\end{smallmatrix}\right]\end{aligned}$$

   (b) Show that $\mathrm{Hom}_R(-,D)$ is left exact. This means (for a contravariant functor) that for any short exact sequence

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \to 0$$

   the following is exact:

$$0 \longrightarrow \mathrm{Hom}_R(C,D) \xrightarrow{\phi^*} \mathrm{Hom}_R(B,D) \xrightarrow{\psi^*} \mathrm{Hom}_R(A,D).$$

   (c) Give an example of an injective R–module map $\phi : M \to N$ such that ϕ^* is not injective, and an example of an injective R-module map $\psi : M \to N$ such that ψ^* is not surjective.
   Remark: An R–module I is called *injective* if $\mathrm{Hom}_R(-,I)$ is exact.

2. In this question, we will study a particularly important instance of the Hom functor. Let k be a field, and let k–$\underline{\mathrm{vect}}$ denote the category of finite dimension k-vector spaces. Define the *dual space functor* by

$$\begin{aligned}k\text{–}\underline{\mathrm{vect}} &\longrightarrow k\text{–}\underline{\mathrm{vect}}\\ V &\longmapsto V^* := \mathrm{Hom}_k(V,k)\end{aligned}$$

   Throughout this question, let A^T or v^T denote the *transpose* of a matrix A or column vector v.
   (a) Let V be a finite dimensional k-vector space. Given a choice of basis $B = \{b_1,\ldots,b_n\}$ for V, define a nondegenerate symmetric bilinear form $(-,-)$ on V such that

$$(b_i,b_j) = \begin{cases} 1, & i = j\\ 0, & i \neq j\end{cases}$$

   (Note that this is slightly different than the inner product defined in Warm-up Problem 15. The form $(-,-)$ has the advantage over $\langle -,-\rangle$ that it is linear in both coordinates, but the disadvantage that it is not positive definite when $k = \mathbb{C}$.)
   Let $v,u \in V$. Show that this definition completely determines the value of (v,u), and moreover that (v,u) is equal to the *dot product* $v^T u$ of v and u when they are expressed with respect to the basis B.
   (b) For each $i = 1,\ldots,n$, define the map $b^i : V \to k$ by

$$b^i(v) := (b_i, v).$$

   Check that b^i is a *functional*, ie, a k–linear map $V \to k$, and show moreover that the map $b_i \mapsto b^i$ extends to a k-linear map

$$\begin{aligned}V &\longrightarrow V^*\\ w &\longmapsto \big[v \mapsto (w,v)\big]\end{aligned}$$

(c) Show that the functionals $b^1, \ldots, b^n$ are linearly independent and span V^*, and therefore form a basis B^* (called the *dual basis* to B). Conclude that a choice of basis for V defines an isomorphism of vector spaces $V \cong V^*$.
Remark: Although V and V^* are isomorphic as abstract vector spaces, they are not *naturally isomorphic* in the sense that any isomorphism involves a choice of basis or choice of bilinear form. There is, however, a natural isomorphism between V and $(V^*)^*$.

(d) Show that if $A : V \to W$ is a linear map given by a matrix with respect to orthonormal bases B_V and B_W. Show that

$$(w, Av)_W = (A^T w, v)_V.$$

Hint: Use the formula $(u, u') = u^T u'$. This should be a one-line solution.

(e) Show that if $A : V \to W$ is a linear map given by a matrix with respect to bases B_V and B_W, then the induced map $W^* \to V^*$ is given by the matrix A^T with respect to the dual bases B_V^* and B_W^*.

(f) Suppose G is a group with a linear action on a k-vector space V given by $\rho : G \to GL(V)$. Then we can construct an associated representation of G on V^*, called the *dual representation* ρ^* of ρ. Define ρ^* by

$$\rho^*(g) : \phi \longmapsto \Big[v \mapsto \phi\Big(\rho(g)^{-1}(v)\Big)\Big] \qquad \forall\, \phi \in V^*,\ g \in G$$

Verify that ρ^* defines a linear representation of G. If A is the matrix representing the action of a group element $g \in G$ on V with respect to a basis B, show that the matrix for g on V^* with respect to B^* is given by $(A^{-1})^T$, the inverse transpose of A.

3. (a) For integers $m, n > 1$, compute the abelian groups $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ and $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$.

   (b) For integer $n > 1$, compute the abelian groups $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ and $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.

   (c) Compute the rational vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.

   (d) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are **not** isomorphic as vector spaces over $\mathbb{R}$.

   (e) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ **are** isomorphic as vector spaces over $\mathbb{Q}$.

   Note: By "compute" an abelian group I mean describe the group in terms of the classification of finitely generated abelian groups, as a product of cyclic groups. By "compute" a vector space I mean determine its dimension.

4. **(The tensor-hom adjunction.)** Let S, R by rings. Let A be an (S, R)–bimodule, B a left R–module, and C a left S–module. Prove that there is a (well-defined) isomorphism of abelian groups

$$\mathrm{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \mathrm{Hom}_R\big(B, \mathrm{Hom}_S(A, C)\big)$$
$$\big[f : a \otimes b \longmapsto f(a \otimes b)\big] \longmapsto \Big[b \longmapsto [a \longmapsto f(a \otimes b)]\Big]$$