Reading: Dummit–Foote Ch 10.4, 10.5, 11.5, 12.1, 12.2. We will not cover the computational algorithms.

Summary of definitions and main results

Definitions we've covered: Rank of a module, free rank, invariant factors, elementary divisors, characteristic polynomial, minimal polynomial, companion matrix, rational canonical form.

Main results: $D \otimes_R -$ is a right-exact covariant functor, how to use the universal property (or right exactness) to compute tensor products in specific examples, fundamental theorem for finitely generated modules over a PID (invariant factor form and elementary divisor form), matrices are classified up to conjugacy by their rational canonical forms.

Warm-Up Questions

1. Use the universal property of the tensor product $\mathbb{Z}/12\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/20\mathbb{Z}$ to verify that $3 \otimes 6$ is nonzero.
2. Which of the following rings are PIDs? Let $\mathbb{F}$ denote a field.

   $\mathbb{F}$, $\mathbb{F}[x]$, $\mathbb{F}[x,y]$, $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}[x]$, $M_n(\mathbb{F})$, division ring, quotient of a PID

3. Compute the torsion submodules of the following:
   (a) A finite abelian group G (as a $\mathbb{Z}$–module)
   (b) $\mathbb{Z}/5\mathbb{Z}$ as a module over $\mathbb{Z}$, and over $\mathbb{Z}/5\mathbb{Z}$
   (c) The $\mathbb{Z}$–modules $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}/\mathbb{Z}$, and $\mathbb{R}/\mathbb{Z}$
   (d) A vector space V over a field $\mathbb{F}$
   (e) A free R–module F
4. Let R be an integral domain.
   (a) Let N be an R–module. Show that if its annihilator $\mathrm{Ann}(N)$ is nonzero, then N is a torsion module.
   (b) Is the converse true? If N is torsion, must its annihilator be nonzero? (You proved on Midterm I that this is true when N is finitely generated.)
5. Suppose that R is a PID and M a finitely generated R–module with invariant factors $a_1, \ldots, a_m$. Show that the annihilator of $\mathrm{Tor}(M)$ is the ideal generated by a_m.
6. Explain why the notions of torsion and linear independence in R–modules are better behaved when R is an integral domain.
7. Let R be an integral domain, and M a finitely generated R–module *Hint:* Dummit–Foote 12.1 Prop. 3.
   (a) Suppose $\{m_1, m_2, \ldots, m_n\}$ generates M. Prove that any linearly independent set in M must have n or fewer elements.
   (b) Show that the *rank* is well-defined for a finitely generated module over an integral domain, in the following sense: If S and T are both finite linearly independent sets M, and each is maximal (in the sense that adding any additional element of M would yield a linearly dependent set), then S and T must have the same cardinality.
8. Let R be an integral domain. Suppose that $\mathbb{F}$ is a field containing R. Show that any linearly independent set $\{m_1, \ldots, m_n\}$ in an R–module M will yield a linearly independent set of vectors $\{1 \otimes m_1, \ldots, 1 \otimes m_n\}$ in the $\mathbb{F}$–vector space $\mathbb{F} \otimes_R M$. Conclude that the $\mathrm{rank}(M) = \dim_{\mathbb{F}}(\mathbb{F} \otimes_R M)$.
   Remark: When R is an integral domain, it is always possible to construct a field $\mathbb{F}$ containing R (its *field of fractions*). The dimension $\dim_{\mathbb{F}}(\mathbb{F} \otimes_R M)$ is sometimes taken as the definition of the rank of M.
9. Let R be an integral domain.

(a) Conclude from Exercise 7 that any set of $(n+1)$ elements in R^n are linearly dependent, and therefore that R^n has rank n.

(b) Prove that any torsion R–module has rank zero.

(c) Show that for any R–module M, rank(M)=rank($M/\text{Tor}(M)$).

10. Find the invariant factors and elementary divisors of the finitely generated abelian group

$$M \cong \mathbb{Z}^{12} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{9\mathbb{Z}} \oplus \frac{\mathbb{Z}}{5\mathbb{Z}} \oplus \frac{\mathbb{Z}}{18\mathbb{Z}} \oplus \frac{\mathbb{Z}}{15\mathbb{Z}}.$$

11. (a) Show that the ideal $I = (2, x) \subseteq R = \mathbb{Z}[x]$ is a finitely generated, torsion-free R–module, but not a free R–module. What is the rank of I?

(b) In contrast, what can you say about finitely generated torsion-free modules over a PID?

12. **(Linear algebra review.)**

(a) 1. Define what it means for two matrices to be *conjugate* (or *similar*)

2. What is the conjugacy class of the zero matrix? The identity matrix? A scalar matrix?

3. Explain why two matrices are conjugate if and only if they represent the same linear map with respect to different bases.

4. Show that conjugate matrices have the same determinant.

5. Show that $(ABA^{-1})^n = AB^nA^{-1}$.

13. **(Linear algebra review.)** Let $A : V \to V$ be a linear map on a finite dimensional vector space V.

(a) Suppose A is a *block diagonal matrix*, ie, it has square matrices $\mathbf{A}_i$ (its *blocks*) on the diagonal:

$$A = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix}$$

$$\left(\text{eg. } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 6 & 4 \end{bmatrix} \text{ has } \mathbf{A}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \text{ and } \mathbf{A}_2 = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{ has } \mathbf{A}_1 = [1], \mathbf{A}_2 = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \text{ and } \mathbf{A}_3 = [4] \right)$$

Explain how the blocks of A correspond to a decomposition of V into a direct sum of subspaces $V = V_1 \oplus \cdots \oplus V_n$ where each V_i is invariant under the action of A. (The matrix A is sometimes called the *direct sum* of its blocks $A = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_n$.)

(b) Conversely, explain why, if V decomposes into a direct sum of subspaces that are invariant under A, then the corresponding matrix for A will be block diagonal. (What are the sizes of the blocks?)

(c) Observe that Trace(A) =Trace($\mathbf{A}_1$) + $\cdots$ +Trace($\mathbf{A}_n$), and Det(A) =Det($\mathbf{A}_1$) $\cdots$ Det($\mathbf{A}_n$).

(d) What is the product of two block diagonal matrices (assuming blocks of the same sizes)?

(e) Show that for any exponent $p \in \mathbb{Z}_{\geq 0}$, the matrix A^p is block diagonal with blocks $\mathbf{A}_1^p, \ldots, \mathbf{A}_m^p$.

14. Let $T : V \to V$ be a linear map. Show that T is the zero map if and only if $Tv = 0$ for all $v \in V$. Conclude in particular that T satisfies a polynomial $p(x)$ if and only if $p(T)v = 0$ for all $v \in V$.

15. (a) Let A be an $n \times n$ matrix. Show that A satisfies its minimal polynomial (ie, $m_A(A)$ is the zero matrix), and that it is the smallest-degree monic polynomial that vanishes at A.

(b) Let T be a linear map. Show that $m_T(A) = 0$ for any matrix representation A of T, and that the minimal polynomial $m_T(x)$ is the smallest-degree polynomial with this property.

(c) Show that the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ satisfies the polynomial $x^2 - x - 2$. Conclude that this polynomial must be in the ideal $(m_T(x))$ and therefore a multiple of the minimal polynomial $m_T(x)$. What are the possibilities for $m_T(x)$?

16. Let A be an $n \times n$ square matrix and B the $2n \times 2n$ block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. Let $c_A(x)$ and $m_A(x)$ be the characteristic and minimal polynomials of A. What are the characteristic and minimal polynomials of B? Observe in particular that the minimal polynomial of B can have degree at most n.

17. Prove that the minimal polynomial of

$$A = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix}$$

is the least common multiple of the minimal polynomials of the blocks $\mathbf{A}_i$.

18. Prove that conjugate matrices have the same characteristic polynomial and the same minimal polynomial.

19. Let $\mathbb{F}$ be a field and all matrices taken over $\mathbb{F}$.
    (a) Show that 2×2 matrices that are not scalar matrices are conjugate iff they have the same characteristic polynomial.
    (b) Show that 3×3 matrices are conjugate iff they have the same characteristic and minimal polynomials.
    (c) Write down two nonconjugate 4×4 matrices with the same minimal and characteristic polynomials.

20. Let $T : V \to V$ be a linear map on a finite dimensional $\mathbb{F}$–vector space V. Describe how to construct an $\mathbb{F}[x]$–module corresponding to T. Explain the relationship between T and this $\mathbb{F}[x]$–module, and explain what we can infer about T from the invariant factor decomposition of this $\mathbb{F}[x]$–module.

21. What is the characteristic polynomial of a companion matrix $\mathcal{C}_{a(x)}$?

22. Suppose that V is an $\mathbb{F}[x]$–module that is n dimensional over the field $\mathbb{F}$. Let T be the linear map on V given by multiplication by x.
    (a) What does the dimension of V imply about the degrees of the invariant factors $a_1(x)$, $a_2(x)$, $\ldots, a_m(x)$? About the degree of the characteristic polynomial of T?
    (b) Show in particular that the minimal polynomial of T has degree at most n.
    (c) If $m_T(x)$ has degree n, what does this tell you about the invariant-factor decomposition of V?
    (d) Conversely, suppose that V is a cyclic $\mathbb{F}[x]$–module. What can you conclude about $m_T(x)$?

23. A linear map $L : V \to V$ is called *nilpotent* if $L^k = 0$ for some positive $k \in \mathbb{Z}$. Show that the following $n \times n$ matrices $J_{0,n}$ are nilpotent, and find the minimal k such that $J_{0,n}^k = 0$.

$$J_{0,1} = \begin{bmatrix} 0 \end{bmatrix} \qquad J_{0,2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad J_{0,3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad J_{0,4} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J_{0,n} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \qquad \begin{bmatrix} J_{0,2} & 0 & 0 \\ 0 & J_{0,2} & 0 \\ 0 & 0 & J_{0,3} \end{bmatrix} \text{ (here 0 denotes the zero matrix).}$$

Assignment Questions

1. **(The functor $D \otimes_R -$ is right exact.)** Let R be any ring, and D a right R–module.
   (a) Show that the following map of categories is well-defined and a covariant functor:

$$D \otimes_R - : R\text{–}\underline{\text{Mod}} \longrightarrow \underline{\text{Ab}}$$
$$M \longmapsto D \otimes_R M$$
$$\{f : M \to N\} \longmapsto \begin{bmatrix} f_* : D \otimes_R M \longrightarrow D \otimes_R N \\ f_*(d \otimes m) = d \otimes f(m) \end{bmatrix}$$

   (b) Show that the functor $D \otimes_R -$ is right exact.

   Hint: Dummit-Foote 10.5 Theorem 39. An alternate argument on p402 uses the Hom-tensor adjunction.

2. For any ring R and right R–module D, the functor $D \otimes_R -$ is right exact. A similar argument shows that for any left R–module D the functor $- \otimes_R D$ is right exact. In this question, we will use the right-exactness of these functors together with a presentation of an R–module M to compute $D \otimes_R M$ or $M \otimes_R D$.
   (a) Use the right-exactness of the functor $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} -$ and the short exact sequence of $\mathbb{Z}$–modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

   to (re)compute $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$.
   (b) More generally, let R be a ring and I a two-sided ideal. Use the right exactness of $- \otimes_R N$ and the short exact sequence of R–modules

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

   to (re)prove the result: $R/I \otimes_R N \cong N/IN$.
   (c) Let k be a field and let $R = k[x, y]$. Give simple descriptions of the following tensor products, and determine their dimensions over k.

$$\frac{R}{(x)} \otimes_R \frac{R}{(x-y)} \qquad \frac{R}{(x)} \otimes_R \frac{R}{(x-1)} \qquad \frac{R}{(y-1)} \otimes_R \frac{R}{(x-y)}$$

3. Let $\mathbb{F}$ be a field (not of characteristic 2) and V a vector space over $\mathbb{F}$ with basis $\{x_1, \ldots, x_n\}$.
   (a) Verify that $\text{Sym}^k(V)$ is a vector space over $\mathbb{F}$ with basis given by the set of monomials in the variables $\{x_1, x_2, \ldots, x_n\}$ of total degree k. (*Remark:* There are $\binom{n+k-1}{n-1}$ such monomials).
   (b) Verify that $\wedge^k V$ is isomorphic to the $\mathbb{F}$–vector space with a basis given by elements of the form $x_{i_1} x_{i_2} \cdots x_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. (*Remark:* There are $\binom{n}{k}$ such elements).
   Hint for (a) and (b): To show these elements are linearly independent, is enough to use the universal property to define a symmetric or alternating multilinear map $V^k \to \mathbb{C}$ that factors through $\text{Sym}^k V$ or $\wedge^k V$ such that it takes value 1 on one basis element and 0 on all others.
   (c) Show that the additive groups

$$T^* V := \bigoplus_{i=0}^{\infty} V^{\otimes i} \qquad \text{Sym}^* V := \bigoplus_{i=0}^{\infty} \text{Sym}^i(V) \qquad \wedge^* V := \bigoplus_{i=0}^{\infty} \wedge^i V$$

   each have a natural ring structure. You do not need to check the axioms for a ring, but define and briefly describe the multiplication in each case, and identify the multiplicative identity. The multiplication on T^*V is called *noncommutative*, the multiplication on $\text{Sym}^* V$ is *commutative*, and the multiplication on $\wedge^* V$ is called *anti-commutative*.

(d) Show that you can identify Sym^*V, and $\wedge^*V$ as **subspaces** of T^*V via the maps

$$x_1 x_2 \cdots x_k \longmapsto \frac{1}{k!}\sum_{\sigma\in S_k}\sigma(x_1\otimes x_2\otimes\cdots\otimes x_k) \qquad \text{and} \qquad x_1\wedge x_2\wedge\cdots\wedge x_k \longmapsto \frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma)\sigma(x_1\otimes x_2\otimes\cdots\otimes x_k)$$

(e) Show that $V\otimes_{\mathbb{F}} V \cong \mathrm{Sym}^2(V)\oplus\wedge^2 V$.

(f) Show that $V\otimes_{\mathbb{F}} V\otimes_{\mathbb{F}} V \supsetneq \mathrm{Sym}^3(V)\oplus\wedge^3 V$.

4. (a) Let H be a subgroup of a group G. Show by example that there may be elements in H which are not conjugate in H, but are conjugate in G. What is the relationship between the conjugacy classes in H and the conjugacy classes in G?

(b) Let $\mathbb{E}$ be a field and $\mathbb{F}$ a subfield of $\mathbb{E}$. Let A and B be $n\times n$ matrices with coefficients in $\mathbb{F}$. Use the theory of rational canonical form to show that A and B are conjugate in $M_n(\mathbb{E})$ if and only if they are conjugate in $M_n(\mathbb{F})$.

Remark. This implies:

- If two matrices A and B are conjugate, then they are conjugate by a matrix with coefficients in the smallest field over which the entries of A and B are defined.
- Matrices that are not conjugate in $M_n(\mathbb{F})$ cannot become conjugate when we extend scalars to a field extension.
- Suppose $\mathbb{F}$ is a field that is not algebraically closed (like $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{F}_q$). Two linear maps over $\mathbb{F}$ are conjugate if and only if they have the same Jordan canonical form (over the algebraic closure of $\mathbb{F}$) – even if their Jordan canonical form is not defined over $\mathbb{F}$.