Reading: Dummit–Foote Ch 12.3, 12.1, 18.1.

Summary of definitions and main results

Definitions we've covered: Jordan canonical form, Jordan block $J_{\lambda,k}$, Smith normal form, simple (or irreducible) module, decomposable module, completely reducible module.

Main results: Existence and uniqueness of Jordan canonical form; Jordan canonical form classifies linear maps up to conjugacy, procedure for putting matrices into Smith Normal Form, proof outline of structure theorem for finitely generated modules over PID (existence, uniqueness), properties of the averaging map.

Warm-Up Questions

1. Define an *algebraically closed* field. What is the *algebraic closure* of a field $\mathbb{F}$?
2. Show that if $\mathbb{F}$ is an algebraically closed field, then the prime elements of $\mathbb{F}[x]$ are polynomials of the form $(x-\lambda)$, $\lambda \in \mathbb{F}$.
3. (a) Consider the $\mathbb{F}[x]$–module $\mathbb{F}[x]/(x-\lambda)^k$ with basis $(x-\lambda)^{k-1}, (x-\lambda)^{k-2}, \ldots, (x-\lambda), 1$. Carefully explain why, with respect to this basis, the map "multiplication by x" acts on this module by the Jordan matrix

$$J_{\lambda,k} = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & & \\ & & & \ddots & 1 \\ & & & & \lambda \end{bmatrix}.$$

   (b) Consider the elementary divisor form of the structure theorem for a finitely generated torsion $\mathbb{F}[x]$–module V:

$$V \cong \frac{\mathbb{F}[x]}{(x-\lambda_1)^{k_1}} \oplus \frac{\mathbb{F}[x]}{(x-\lambda_2)^{k_2}} \oplus \cdots \oplus \frac{\mathbb{F}[x]}{(x-\lambda_\ell)^{k_\ell}}$$

   Conclude that, in a suitably chosen basis, the map "multiplication by x" will correspond to the block diagonal matrix:

$$\begin{bmatrix} J_{\lambda_1,k_1} & & & \\ & J_{\lambda_2,k_2} & & \\ & & \ddots & \\ & & & J_{\lambda_\ell,k_\ell} \end{bmatrix}$$

   (c) Explain why the uniqueness of the elementary divisor decomposition of V implies that (up to re-ordering of the blocks) this will be the only matrix in Jordan canonical form that represents the map "multiplication by x" in any basis.

   (d) Explain why two linear maps are conjugate if and only if they have the same Jordan canonical form.
4. Find the characteristic polynomial and the minimal polynomials of the following matrices.

$$\begin{pmatrix} 3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3 \end{pmatrix} \quad \begin{pmatrix} 3&1&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3 \end{pmatrix} \quad \begin{pmatrix} 3&1&0&0\\0&3&0&0\\0&0&3&1\\0&0&0&3 \end{pmatrix} \quad \begin{pmatrix} 3&1&0&0\\0&3&1&0\\0&0&3&0\\0&0&0&3 \end{pmatrix} \quad \begin{pmatrix} 3&1&0&0\\0&3&1&0\\0&0&3&1\\0&0&0&3 \end{pmatrix}$$

5. Figure out how to use your favourite mathematics software program to output the rational canonical form and Jordan canonical form a matrix. (For example, try typing the following into Wolfram alpha:

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jordan canonical form {{5, 4, 2, 1}, {0, 1, -1, -1}, {-1, -1, 3, 0}, {1, 1, -1, 2}} ).
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(This exercise is optional for this course, but a good mathematical tool to have for the future. Algorithms for putting matrices in Rational and Jordan Canonical Form are described in Dummit–Foote Ch. 12, but these are also beyond the scope of our course.)

6. For each of the following $\mathbb{C}[x]$–modules, list the invariant factors, the elementary divisors, and write the rational canonical form and Jordan canonical form of the linear map "multiplication by x". State the minimal and characteristic polynomials.
   (a) $V \cong \dfrac{\mathbb{C}[x]}{(x-1)^2} \oplus \dfrac{\mathbb{C}[x]}{(x-1)(x-2)}$
   (b) $V \cong \dfrac{\mathbb{C}[x]}{(x-1)(x-2)(x-3)}$
   (c) $V \cong \dfrac{\mathbb{C}[x]}{(x-1)} \oplus \dfrac{\mathbb{C}[x]}{(x-1)^2} \oplus \dfrac{\mathbb{C}[x]}{(x-1)^2}$

7. Compute the Jordan canonical form of the following matrices:

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

8. Determine all possible Jordan canonical forms for linear maps with characteristic polynomial

$$(x-1)^3(x-2)^2.$$

9. (a) Suppose a complex matrix A satisfies the equation $A^2 = -2A - 1$. What are the possibilities for its Jordan canonical form?
   (b) Suppose a complex matrix A satisfies $A^3 = A$. Show that A is diagonalizable. Would this result hold if A had entries in a field of characteristic 2?

10. Prove that an $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

11. Suppose that a polynomial $f(x) \in \mathbb{C}[x]$ has no repeated roots. Show that all linear maps with characteristic polynomial $f(x)$ are similar.

12. Fix $\lambda \in \mathbb{F}$. Show that the number of conjugacy classes of matrices with characteristic polynomial $(x-\lambda)^n$ is equal to the number of partitions of n.

13. **(Linear Algebra Review).** Consider the set of $m \times n$ matrix over $\mathbb{R}$.
   (a) Define *row operations* on $m \times n$ matrices, and define the *reduced row echelon form (RREF)* of a matrix M.
   (b) Find the RREF of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 0 & 4 \\ -1 & 2 & 2 \end{bmatrix}$.
   (c) Show that row operations can be represented by left multiplication by *elementary matrices*.
   (d) Show that elementary matrices generate $\mathrm{GL}_m(\mathbb{R})$. (*Hint:* Show matrices in $\mathrm{GL}_m(\mathbb{R})$ row reduce to the identity.)
   (e) Conclude that row operations correspond to the action of $\mathrm{GL}_m(\mathbb{R})$ on $M_{m\times n}(\mathbb{R})$ on the left, and that RREF form of a matrix M is a canonical representative for its $\mathrm{GL}_m(\mathbb{R})$-orbit.
   (f) Suppose instead we were acting on $M_{m\times n}(\mathbb{Z})$ by $\mathrm{GL}_m(\mathbb{Z})$. Which operations are possible? Explain the relationship to the row operations performed to put matrices in *Smith normal form.*

14. Let R be a Euclidean domain and M a finitely generated submodule. To prove the invariant factor decomposition for M, we first constructed a surjection φ

$$0 \longrightarrow \ker(\varphi) \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$$

and then computed the *Smith normal form* of the $m \times n$ *relations matrix* between a basis $x_1, \dots, x_n$ for R^n and a generating set $y_1, \dots y_m$ for $\ker(\varphi)$.

(a) Define the relations matrix. Explain the sense in which the rows of the relations matrix correspond to generators y_i of $\ker(\varphi)$, and the columns of the matrix correspond to basis elements x_j of R^n.

(b) Explain how column operations on the relations matrix correspond to operations on the basis $\{x_j\}$, and how row operations correspond to operations on the generators $\{y_i\}$.

(c) Verify that for each row and column operation, the modified set $\{x_j\}$ will still be a basis for R^n, and the modified set $\{y_i\}$ will still be a generating set for $\ker(\varphi)$.

(d) Describe what it means for the relations matrix to be in *Smith normal form*, and the structure of the basis for R^n and generating set of $\ker(\varphi)$ constructed in the process of putting the matrix in this form.

(e) Explain how to compute the invariant factor decomposition of $M \cong R^n/\ker(\varphi)$ from the Smith normal form of the matrix.

15. Assume the same set up as in the previous question, with $R = \mathbb{Z}$.

(a) For each of the matrices in Smith normal form, concretely describe the short exact sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow \mathbb{Z}^n \xrightarrow{\varphi} M \longrightarrow 0$$

and state the free rank and invariant factors for M.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(b) Explain how the rows of zeroes in matrices A and D correspond to a redundancy in our choice of generators $\{y_1, \dots, y_m\}$ for the kernel $\ker(\varphi)$.

(c) Explain how the column of zeroes in matrices A and B corresponds to the free part of M.

(d) Explain how the unit 1 in E corresponds to a redundancy in our choice of generators $\{\varphi(x_1), \dots, \varphi(x_n)\}$ for M.

16. Let $\rho : G \to GL(V)$ be a representation of a finite group G, and let $V \cong U \oplus W$ be a decomposition of V into G-invariant subspaces. Show that, in a suitably chosen basis for V, every matrix $\rho(g)$ will be block diagonal, with a block acting on U and a block acting on V.

17. Given representations $G \to GL(V)$ and $G \to GL(U)$, construct a representations $G \to GL(V \oplus U)$ and $G \to GL(V \otimes U)$.

18. Let V be a finite dimensional $\mathbb{F}[x]$–module where x acts by linear map T. Describe how to use the Jordan canonical form of T to determine whether V is simple, reducible, decomposable, indecomposable, or completely reducible.

19. Let V be a representation of a group G over $\mathbb{F}$. Given any covariant functor $\mathcal{F} : \mathrm{Vect}_{\mathbb{F}} \to \mathrm{Vect}_{\mathbb{F}}$ explain why $\mathcal{F}(V)$ has an induced structure of a representation of G.

Assignment Questions

1. (a) Prove that a linear map on a finite dimensional $\mathbb{F}$–vector space is diagonalizable over $\mathbb{F}$ if and only if its minimal polynomial has distinct roots, all contained in $\mathbb{F}$.

   (b) Let $P : V \to V$ be a projection matrix. (This means that there is some decomposition $V = W \oplus U$ such that $P(w + u) = w$ for all $u \in U, w \in W$). Show that $P^2 v = Pv$ for all $v \in V$, and use this relation to show that P is diagonalizable. What are its eigenvalues?

   (c) Let A be an $n \times n$ invertible complex matrix. Show that if A has finite order p then A is diagonalizable, and describe the possible eigenvalues of A.

   (d) Suppose that V is a finite dimensional vector space over $\mathbb{F}$, and $T : V \to V$ is a diagonalizable linear map. Show that the restriction of T to any T–invariant subspace $W \subseteq V$ will also be diagonalizable, and therefore W must be a direct sum of eigenspaces of T.

2. Let $T : V \to V$ be a linear map on a n-dimensional $\mathbb{F}$–vector space V. Let $e_1, \ldots, e_n$ be a basis for V corresponding to the Jordan canonical form of T. Let I denote the identity matrix.

   Recall that an *eigenvector* v of T with *eigenvalue* λ is defined to be a nonzero element of $\ker(\lambda I - T)$, and that the *eigenspace* E_λ is defined to be the subspace of V

   $$E_\lambda = \ker(\lambda I - T) = \{\text{eigenvectors of } T \text{ with eigenvalue } \lambda\} \cup \{0\}$$

   For an eigenvalue λ of T, define the *algebraic multiplicity* of λ to be the multiplicity of the root $(x - \lambda)$ in the characteristic polynomial of T, and the *geometric multiplicity* to be the $\dim_{\mathbb{F}}(E_\lambda)$.

   (a) Let $J_{\lambda,k}$ denote the $k \times k$ Jordan block with diagonal entry λ. Prove that the characteristic polynomial and minimal polynomial of $J_{\lambda,k}$ are both equal to $(x - \lambda)^k$.

   (b) Prove that $J_{\lambda,k}$ has a single one-dimensional eigenspace $E_\lambda = \langle e_1 \rangle$.

   (c) For any linear map T with eigenvalue λ, show that the geometric multiplicity of λ – the dimension of the eigenspace E_λ – is equal to the number of Jordan blocks with diagonal entry λ in the Jordan canonical form of T.

   (d) Let λ be an eigenvector of T. Define the *generalized eigenspace of* λ to be the subspace

   $$G_\lambda = \{v \mid (\lambda I - T)^k v = 0 \text{ for some integer } k > 0\}$$

   (e) Show (in a sentence) that $E_\lambda \subseteq G_\lambda$.

   (f) Show that the generalized eigenspace G_λ of V is precisely the direct sum of submodules of the form $\mathbb{F}[x]/(x - \lambda)^\alpha$ in the elementary divisor form of the decomposition of V (called the $(x - \lambda)$-*primary component* of V), by analyzing the action of x on this decomposition.

   (g) Re-prove part (f), this time by analyzing the Jordan canonical form of T. Show that the generalized eigenspace for λ is precisely the subspace of V corresponding to the Jordan blocks with diagonal entry λ. Start by observing that a Jordan block $J_{\mu,k}$ will satisfy the polynomial $(\lambda I - J_{\mu,k})^m$ for some m if and only if $\mu = \lambda$.

   (h) Conclude that V decomposes into a direct sum of generalized eigenspaces for T, and that the algebraic multiplicity of an eigenvalue λ is equal to sum of the sizes of the corresponding Jordan blocks, which is equal to the dimension of G_λ.

   (i) Note as a corollary that dimension of the eigenspace E_λ is no greater than the algebraic multiplicity of λ. Under what conditions are they equal?

3. State instructions for how to read off the following data from the Jordan canonical form of a linear map T, and state each for the specific map T_0 given below. **You do not need justify your computations or instructions.**

$$T_0 = \begin{bmatrix} 2 & 1 & & & & & & & & \\ & 2 & & & & & & & & \\ & & 2 & 1 & & & & & & \\ & & & 2 & & & & & & \\ & & & & 2 & 1 & & & & \\ & & & & & 2 & & & & \\ & & & & & & 2 & & & \\ & & & & & & & 2 & & \\ & & & & & & & & 3 & 1 \\ & & & & & & & & & 3 \end{bmatrix}$$

(a) The eigenvalues of T (with algebraic multiplicity).

(b) The determinant of T.

(c) The characteristic polynomial of T.

(d) The minimal polynomial of T.

(e) The elementary divisors of T.

(f) The invariant factors of T.

(g) The dimension of the eigenspace E_λ associated to any eigenvalue λ of T.

(h) The eigenvalues of T (with geometric multiplicity).

(i) The dimension of the generalized eigenspace G_λ for any eigenvalue λ of T.

4. In class, we sketched a proof of the structure theorem for finitely generated modules over a Euclidean domain R. Reference: Dummit–Foote Ch 12.1 Exercises 17–19.

(a) Let K be a submodule of R^n. Since R^n is Noetherian, K is finitely generated. Explain why K is a free R–module of rank at most n, and that there exists a basis $x_1, \dots, x_n$ for R^n such that K is free on the set $a_1x_1, a_2x_2, \dots, a_kx_k$ for some nonzero $a_i \in R$ satisfying $a_1|a_2|\cdots|a_k$.
(You do not need to re-prove the arguments from the class, you can just quote the conclusion concerning the relations matrix in Smith normal form).
Remark: This property holds for all PID's. Those interested should refer to 12.1 Theorem 4.

(b) Let $\mathbb{Z}^4$ be the free abelian group on the standard basis

$$e_1 = (1,0,0,0), \quad e_2 = (0,1,0,0), \quad e_3 = (0,0,1,0), \quad e_4 = (0,0,0,1).$$

Let M be the submodule generated by the elements

$$M = \left\langle \begin{bmatrix} 4 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 8 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ -2 \end{bmatrix} \right\rangle$$

Find bases for $\mathbb{Z}^4$ and M as described in part (a), by performing row and column operations to put an appropriately defined matrix into Smith normal form. (You can optionally use computer software to do these computations, but include a print-out of your computer work).

(c) What is the invariant factor decomposition for $\mathbb{Z}^4/M$?