Reading: Dummit–Foote 18.1, 18.3, Fulton–Harris "Representation Theory: A first course", Ch 1.1–2.1.

Summary of definitions and main results

Definitions we've covered: V^G, class function, character, character table, the inner product $\langle -, - \rangle_G$

Main results: Schur's lemma; Maschke's theorem; induced $\mathbb{F}[G]$–modules structures on $V \oplus W$, $\mathrm{Hom}_{\mathbb{F}}(V, W)$, V^*, $V \otimes W$, $\wedge^k V$, $\mathrm{Sym}^k(V)$; linear independence of characters

Warm-Up Questions

1. Given an example of a ring R and an R–module M that is:
   (a) irreducible
   (b) reducible, but not decomposable
   (c) decomposable, but not completely reducible
   (d) completely reducible, but not irreducible

2. Let V be a representation of a group G, and recall that V^G denotes the set of vectors in V that are fixed pointwise by the action of every group element $g \in G$. Verify that V^G is a linear subspace of V.

3. Let V and W be representations of a group G over a field k. Define the induced action of G on the k–vector space $\mathrm{Hom}_k(V, W)$, and verify that it satisfies the definition of a representation of G.

4. Complete our proof of Maschke's Theorem: Show that if $\pi_0 : V \to U$ is a projection map (in that π_0 restricts to the identity on $U \subseteq V$), then the map $\pi = \frac{1}{|G|} \sum_{g \in G} g \pi_0 g^{-1}$ is also a projection $V \to U$.

5. (a) Let $\mathbb{C}^n = \langle e_1, \ldots, e_n \rangle$ be the canonical representation of the symmetric group S_n by signed permutation matrices. Explicitly describe the action of the *averaging map* on $\mathbb{C}^n$:
$$\begin{aligned} \psi_{av} : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ v &\longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot v \end{aligned}$$
   (b) Suppose v is an element of the standard subrepresentation $\underline{\mathrm{Std}} = \{a_1 e_1 + \cdots + a_n e_n \mid \sum a_i = 0\}$. What is $\psi_{av}(v)$? *Hint:* First check $\psi_{av}(v)$ on the basis vectors $v = (e_1 - e_i)$ for $\underline{\mathrm{Std}}$.
   (c) Interpret your answer to the previous question, given that we know $\psi_{av} : V \to V$ is a linear projection onto V^G.

6. Let G be a finite group and $\phi : G \to GL(V)$ a G–representation over a field $\mathbb{F}$ with character $\chi_V : G \to \mathbb{F}$. Prove that if V is 1-dimensional, then $\chi_V = \phi$. Show by example that if V is at least 2 dimensional, χ_V may not be a group homomorphism.

7. Recall the character table for the complex representations of the symmetric group S_3.

	$(\bullet)(\bullet)(\bullet)$	$(\bullet\bullet)(\bullet)$	$(\bullet\bullet\bullet)$
$\underline{\mathrm{Trv}}$	1	1	1
$\underline{\mathrm{Alt}}$	1	-1	1
$\underline{\mathrm{Std}}$	2	0	-1

   (a) Let $\mathbb{C}^3$ denote the canonical permutation representation of S_3. Compute the characters of $\mathrm{Sym}^2\mathbb{C}^3$ and $\underline{\mathrm{Alt}} \otimes_{\mathbb{C}} \mathrm{Sym}^2\mathbb{C}^3$.
   (b) Use the character table to decompose $\mathrm{Sym}^2\mathbb{C}^3$ and $\underline{\mathrm{Alt}} \otimes_{\mathbb{C}} \mathrm{Sym}^2\mathbb{C}^3$ as a sum of irreducible representations (in the sense of finding the multiplicity of each irreducible representation in the decomposition).

8. Let V be a finite dimensional vector space over $\mathbb{C}$. Recall the definition and properties of a *(Hermitian) inner product* $\langle -,- \rangle : V \times V \to \mathbb{C}$ on V from the Warm-Up Problems on Homework #6.
   Let $e_1, e_2, \ldots, e_n$ be an *orthonormal* basis V with respect to the inner product.
   (a) Let $v = a_1e_1 + \cdots + a_ne_n$ be an element of V. Show that
   $$\langle v, e_i \rangle = a_i \qquad \text{and} \qquad \langle v, v \rangle = |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2.$$
   (b) Suppose that $v = a_1e_1 + \cdots + a_ne_n$ for **nonnegative integer** coefficients a_i. Show that
   $$\langle v, v \rangle = a_1^2 + a_2^2 + \cdots + a_n^2,$$
   and conclude that $\langle v, v \rangle = 1$ if and only if $v = e_i$ for some i.
   (c) Suppose you have a function $\langle -,- \rangle : V \times V \to \mathbb{C}$ which you know satisfies the conjugate-symmetry and linearity properties of an inner product. Show that, if V has an basis that is orthonormal with respect to the function, then it must be positive definite.

Assignment Questions

1. Let G be a finite **abelian** group, and V a finite-dimensional complex representation of G. Show that V decomposes into a direct sum of 1-dimensional G–representations. Conclude that the image of G in $GL(V)$ is *simultaneously diagonalizable*, that is, there is some basis for V with respect to which every matrix is diagonal.

2. Let G be a finite group and $\mathbb{F}$ a field.
   (a) Suppose that A and B are finite order (therefore diagonalizable) endomorphisms of finite dimensional vector spaces V and W over an algebraically closed field $\mathbb{F}$. Show that the trace of $A \otimes B$ on $V \otimes_{\mathbb{F}} W$ is the product Trace(A)Trace(B).
   Remark: This result also holds when A and B are not diagonalizable, and can be proven (with a little more effort) by considering the bases for V and W putting A and B into Jordan canonical form. It can also be proven for arbitrary fields, using extension of scalars to the algebraic closure.
   (b) Let V and W be finite-dimensional representations of G over an algebraically closed field $\mathbb{F}$. Conclude that the character $\chi_{V \otimes_{\mathbb{F}} W}(g) = \chi_V(g)\chi_W(g)$ for all $g \in G$.
   (c) Let $\phi : G \to \mathrm{GL}(V)$ a finite dimensional representation of G over $\mathbb{C}$. Show that for every $g \in G$, we have $\lambda^{-1} = \overline{\lambda}$ for all eigenvalues λ of $\phi(g)$. *Hint:* The element g has finite order.
   (d) Let V be a finite dimensional representation of G over $\mathbb{C}$, and V^* its dual. Prove that $\chi_{V^*}(g) = \overline{\chi_V(g)}$ for all $g \in G$. (You may quote properties of matrix transposes without proof).
   (e) Let $\mathbb{F}$ be any field, and again let V and W be finite-dimensional representations of G over a field $\mathbb{F}$. Construct an isomorphism of G–representations $\mathrm{Hom}_{\mathbb{F}}(V, W) \cong V^* \otimes_{\mathbb{F}} W$. This isomorphism should be *natural*, that is, it should not require a choice of basis for V or W.
   (f) Suppose $\mathbb{F} = \mathbb{C}$. Show that the character of $\mathrm{Hom}_{\mathbb{C}}(V, W)$ is $\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g)$.
   Remark: This will be a key result in our development of character theory!

3. Let G be a finite group. In this question we will consider finite-dimensional complex G–representations.
   (a) Let $\{V_i\}$ be a finite set of irreducible G–representations. Let $U = \bigoplus V_i^{\oplus a_i}$ and let $W = \bigoplus V_j^{\oplus b_j}$ for $a_i, b_j \in \mathbb{Z}_{\geq 0}$. Compute $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(U, W)$. *Hint:* Homework #4 Question 1 and Schur's Lemma.
   (b) Show that $\langle \chi_W, \chi_U \rangle_G := \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(U, W)$ extends to a Hermitian inner product on the $\mathbb{C}$-vector space spanned by the characters of G (under pointwise scalar multiplication and addition). (We will later show that the characters span the whole space of $\mathbb{C}$–valued class functions on G.)
   (c) Show that the characters of irreducible representations are orthonormal.
   (d) Conclude that the characters of irreducible representations are linearly independent.
   (e) Conclude that V is an irreducible representation if and only if $\langle \chi_V, \chi_V \rangle_G = 1$.