Midterm Exam II

Math 122
25 May 2016
Jenny Wilson

Name: ______________________________

Instructions: This exam has 3 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless directed otherwise. You may cite any results from class or the homeworks without proof, but do give a complete statement of the result you are using.

You have 50 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	9	
2	5	
3	6	
Total:	20	

1. (2 points) (a) Simplify the following. **No justification needed.**

$$\left(\mathbb{Q}[x] \oplus \frac{\mathbb{Q}[x]}{(x-1)} \oplus \frac{\mathbb{Q}[x]}{((x-1)^2)}\right) \otimes_{\mathbb{Q}[x]} \left(\frac{\mathbb{Q}[x]}{(x-1)} \oplus \frac{\mathbb{Q}[x]}{(x-2)}\right)$$

Our solution will use the following three results from class:

- Tensor product distributes over direct sums.
- For any ring R and R–module M, $R \otimes_R M \cong M$.
- For a PID R, $\dfrac{R}{(a)} \otimes_R \dfrac{R}{(b)} \cong \dfrac{R}{\left(\gcd(a,b)\right)}$.

This means

$$\begin{aligned}
\mathbb{Q}[x] \otimes_{\mathbb{Q}[x]} \frac{\mathbb{Q}[x]}{(x-a)} &\cong \frac{\mathbb{Q}[x]}{(x-a)} && \forall a \in \mathbb{Q} \\
\frac{\mathbb{Q}[x]}{((x-1)^m)} \otimes_{\mathbb{Q}[x]} \frac{\mathbb{Q}[x]}{(x-1)} &\cong \frac{\mathbb{Q}[x]}{(x-1)} && \forall m \in \mathbb{Z}_{\geq 1} \\
\frac{\mathbb{Q}[x]}{((x-1)^m)} \otimes_{\mathbb{Q}[x]} \frac{\mathbb{Q}[x]}{(x-2)} &\cong \frac{\mathbb{Q}[x]}{(1)} \cong 0 && \forall m \in \mathbb{Z}_{\geq 1}
\end{aligned}$$

We find:

$$\begin{aligned}
&\left(\mathbb{Q}[x] \oplus \frac{\mathbb{Q}[x]}{(x-1)} \oplus \frac{\mathbb{Q}[x]}{((x-1)^2)}\right) \otimes_{\mathbb{Q}[x]} \left(\frac{\mathbb{Q}[x]}{(x-1)} \oplus \frac{\mathbb{Q}[x]}{(x-2)}\right) \\
&= \left(\mathbb{Q}[x] \otimes_{\mathbb{Q}[x]} \left(\frac{\mathbb{Q}[x]}{(x-1)} \oplus \frac{\mathbb{Q}[x]}{(x-2)}\right)\right) \oplus \left(\left(\frac{\mathbb{Q}[x]}{(x-1)} \oplus \frac{\mathbb{Q}[x]}{((x-1)^2)}\right) \otimes_{\mathbb{Q}[x]} \frac{\mathbb{Q}[x]}{(x-1)}\right) \\
&\qquad \oplus \left(\left(\frac{\mathbb{Q}[x]}{(x-1)} \oplus \frac{\mathbb{Q}[x]}{((x-1)^2)}\right) \otimes_{\mathbb{Q}[x]} \frac{\mathbb{Q}[x]}{(x-2)}\right) \\
&= \left(\frac{\mathbb{Q}[x]}{(x-1)} \oplus \frac{\mathbb{Q}[x]}{(x-2)}\right) \oplus \left(\frac{\mathbb{Q}[x]}{(x-1)} \oplus \frac{\mathbb{Q}[x]}{(x-1)}\right) \oplus 0 \\
&= \left(\frac{\mathbb{Q}[x]}{(x-1)}\right)^{\oplus 3} \oplus \frac{\mathbb{Q}[x]}{(x-2)}
\end{aligned}$$

The solution is $\left(\dfrac{\mathbb{Q}[x]}{(x-1)}\right)^{\oplus 3} \oplus \dfrac{\mathbb{Q}[x]}{(x-2)}$.

(b) (4 points) Compute the following. **No justification needed.**

Let I denote the identity map. A certain linear transformation $T : \mathbb{C}^{10} \to \mathbb{C}^{10}$ satisfies the following:

$$\dim\ker(T-2I)=5 \qquad \dim\ker(T-2I)^2=8 \qquad \dim\ker(T-2I)^3=9$$
$$\dim\ker(T-3I)=1$$

i. State Jordan canonical form for T.

The equation $\dim\ker(T-2I)^3 = 9$ shows that the generalized eigenspace G_2 has dimension at least 9, and $\dim\ker(T-3I)=1$ shows that $\dim G_3$ is at least 1. These observations give the decomposition of $\mathbb{C}^{10}$ into generalized eigenspaces of T.
The equation $\dim\ker(T-2I)=5$ implies that there are five Jordan blocks with eigenvalue $\lambda = 2$. The equation $\dim\ker(T-2I)^2 = 8$ implies that there are $8-5=3$ blocks of size at least two, and the equation $\dim\ker(T-2I)^3=9$ implies that there are $9-8=1$ block of size at most three.

We conclude that the Jordan canonical form for T is

$$\begin{bmatrix}
2 & & & & & & & & & \\
 & 2 & & & & & & & & \\
 & & 2 & 1 & & & & & & \\
 & & & 2 & & & & & & \\
 & & & & 2 & 1 & & & & \\
 & & & & & 2 & & & & \\
 & & & & & & 2 & 1 & & \\
 & & & & & & & 2 & 1 & \\
 & & & & & & & & 2 & \\
 & & & & & & & & & 3
\end{bmatrix}$$

ii. State the invariant factors of the $\mathbb{C}[x]$–module $\mathbb{C}^{10}$ where x acts by the map T.

The invariant factors are

$$(x-2), \qquad (x-2), \qquad (x-2)^2, \qquad (x-2)^2, \qquad (x-3)(x-2)^3$$

(c) (3 points) Complete the following. **No justification needed.**

Let S_3 and A_3 denote the symmetric group and alternating group on 3 letters, respectively. The symmetric group S_3 acts on the set of cosets

$$S_3/A_3 = \{A_3,\ (1\,2)A_3\}$$

by left multiplication. Let $V \cong \mathbb{C}^2$ be the $\mathbb{C}$–vector space with basis S_3/A_3, with the induced action of S_3.

i. Compute the character of V, and add it to the following character table.

The identity element acts on V by fixing both basis elements A_3, $(1\,2)A_3$. Its trace is 2. Each 2–cycle acts by interchanging the two basis elements. Their trace is zero. The 3–cycles also act by fixing both basis elements; their trace is 2.

	$(\bullet)(\bullet)(\bullet)$	$(\bullet\bullet)(\bullet)$	$(\bullet\bullet\bullet)$
Trv	1	1	1
Alt	1	-1	1
Std	2	0	-1
V	2	0	2

ii. Find the decomposition of V as a direct sum of irreducible S_3–representations (ie, determine the multiplicities of Trv, Alt, and Std in V).

There are four 2–dimensional S_3–representations:

Std, Trv⊕Trv, Alt⊕Alt, and Trv⊕Alt.

By inspection, $\chi_V = \chi_{\text{Trv}} + \chi_{\text{Alt}}$ and so $V =$ Trv⊕Alt.

2. Suppose that R is a **commutative** ring with the property that every submodule of a free R–module is free.

   (a) (2 points) Prove that R is a domain, ie, R has no zero divisors.

   We prove the contrapositive. Suppose that R were not a domain, so there are some nonzero elements $a, b \in R$ with $ab = 0$. Then the submodule $bR \subseteq R$ is not free: given any element $bx \in bR$,

   $$a(bx) = (ab)x = 0x = 0$$

   is a nontrivial R–linear dependency. The ideal bR is nonzero (it contains the element b), but a maximal linearly independent subset is the empty set, so bR is not free.

   (b) (3 points) Prove that R is a principal ideal domain.

   Let $I \subseteq R$ be any nonzero ideal. Viewing I as an R–submodule of the free rank–1 R–module, the assumption implies that I must be free. Let $B \subseteq I$ be a basis.

   Given any two distinct elements $b_1, b_2 \in I$, then

   $$(b_2)(b_1) + (-b_1)(b_2) = 0$$

   is a nontrivial R–linear dependency between b_1 and b_2. Hence, any two elements of I are linearly dependent, and the basis B must have at most 1 element. We conclude that I is principally generated by the single element set B, and R is a principal ideal domain.

3. (6 points) Let R be a PID. Prove that there is an isomorphism of R–modules

$$\mathrm{Hom}_R\left(\frac{R}{(a)}, \frac{R}{(b)}\right) \cong \frac{R}{(d)} \qquad \text{where } d = \gcd(a, b).$$

You can likely solve this problem using whatever argument you used for Homework #6 Problem 3(a) in the case $R = \mathbb{Z}$. Here is one argument:

Consider the set of R–module maps $\Phi : R \to \frac{R}{(b)}$. We proved on the homework that Φ is determined by the image $\Phi(1)$, and $\Phi(1)$ can be any element of $\frac{R}{(b)}$. By the Factor Theorem, Φ factors through $\frac{R}{(a)}$ if and only if $(a) \subseteq \ker(\Phi)$, that is, if $\Phi(a) = a\Phi(1) = 0$. And, in this case, Φ factors through uniquely.

Let $\Phi(1) = m + (b)$. The element $a\Phi(1) = 0$ in $\frac{R}{(b)}$ if and only if $b|am$, which holds if and only if $\frac{b}{d}\Big|\left(\frac{a}{d}m\right)$. Since $\frac{b}{d}$ and $\frac{a}{d}$ are coprime, this holds if and only if $\frac{b}{d}\Big|m$. Thus, Φ factors through $\frac{R}{(a)}$ if and only $\Phi(1)$ is contained in the submodule $\left(\frac{b}{d}\right)\frac{R}{(b)}$ of $\frac{R}{(b)}$.

The above argument implies that there is a bijective map

$$\begin{aligned} \mathrm{Hom}_R\left(\frac{R}{(a)}, \frac{R}{(b)}\right) &\longrightarrow \left(\frac{b}{d}\right)\frac{R}{(b)} \\ \phi &\longmapsto \phi(1) \end{aligned}$$

Since $(r\phi + \psi)(1) = r\Big(\phi(1)\Big) + \psi(1)$ for all $r \in R$ and maps ϕ and ψ, this bijection is an R–module map and hence an R–module isomorphism.

To conclude the result, it remains to show that $\left(\frac{b}{d}\right)\frac{R}{(b)}$ is isomorphic to $\frac{R}{(d)}$. Consider the surjective R–module map

$$\begin{aligned} R &\longrightarrow \left(\frac{b}{d}\right)\frac{R}{(b)} \\ 1 &\longmapsto \frac{b}{d} \end{aligned}$$

An element r is in the kernel of this map if and only if $\frac{rb}{d}$ is zero in $\frac{R}{(b)}$, which is true if and only if $b\Big|\frac{rb}{d}$. This holds if and only if $d|r$. Hence the kernel of the map is the ideal (d), and by the first isomorphism theorem this map defines an isomorphism

$$\frac{R}{(d)} \xrightarrow{\cong} \left(\frac{b}{d}\right)\frac{R}{(b)} \qquad \text{as desired.}$$