Midterm Exam I

Math 122
27 April 2016
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Name: ______________________________

Instructions: This exam has 3 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless directed otherwise. You may cite any results from class or the homeworks without proof, but do give a complete statement of the result you are using.

You have 50 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	5	
2	9	
3	6	
Total:	20	

1. (a) (2 points) Consider the $\mathbb{Z}$–module $M = \mathbb{Q}/\mathbb{Z}$. Compute the torsion submodule $\mathrm{Tor}(M)$ and the annihilator $\mathrm{ann}(M)$ of M in $\mathbb{Z}$. **No justification necessary.**

   $\mathrm{Tor}(M) = \mathbb{Q}/\mathbb{Z}$

   By definition $\mathrm{Tor}(M) = \{m \in M \mid nm = 0 \text{ for some nonzero } n \in \mathbb{Z}\}$. In the case of $\mathbb{Q}/\mathbb{Z}$, every element is torsion. If $m \in \mathbb{Q}/\mathbb{Z}$ is an element with representative $\frac{a}{b} \in \mathbb{Q}$, then m is annihilated by $b \in \mathbb{Z}$.

   $\mathrm{ann}(M) = \{0\}$

   By definition, $\mathrm{ann}(M) = \{n \in \mathbb{Z} \mid nm = 0 \text{ for every } m \text{ in M}\}$. Notice, however, that if we take any nonzero integer $n \in \mathbb{Z}$, then (for example) if we take $m = \dfrac{1}{n+1}$, we find that $nm = \left(\dfrac{n}{n+1}\right)$ is nonzero in $\mathbb{Q}/\mathbb{Z}$, so n is not contained in $\mathrm{ann}(M)$.

   (b) (3 points) Let R be an integral domain and N an R–module. Show that if N is a finitely generated torsion module, then $\mathrm{ann}(N) \subseteq R$ is nonzero.

   Suppose that N is finitely generated by the elements $n_1, n_2, \dots, n_k$. Since N is a torsion module, for each i there exists some nonzero $r_i \in R$ such that $r_i n_i = 0$. Consider the product $r := r_1 r_2 \cdots r_k$. Since R is an integral domain, this product is nonzero. Moreover, since R is commutative, given any generator n_i,

$$\begin{aligned} rn_i &= (r_1 r_2 \cdots r_k)n_i \\ &= (r_1 r_2 \cdots r_{i-1} r_{i+1} \cdots r_k r_i)n_i \\ &= r_1 r_2 \cdots r_{i-1} r_{i+1} \cdots r_k (r_i n_i) \\ &= 0. \end{aligned}$$

   Given any element $n \in N$, we may write n as an R–linear combination of the generators

$$n = a_1 n_1 + \cdots + a_k n_k \qquad a_i \in R$$

   hence

$$\begin{aligned} rn &= r(a_1 n_1 + \cdots + a_k n_k) \\ &= a_1(rn_1) + \cdots + a_k(rn_k) \\ &= 0 + \cdots + 0 \\ &= 0 \end{aligned}$$

   Therefore r annihilates every $n \in N$, and we conclude that r is a nonzero element in $\mathrm{ann}(N)$.

2. Consider the group $G = \mathbb{Z}$, and the module $V \cong \mathbb{Q}^2$ over the rational group ring $\mathbb{Q}[G]$ defined by the map

$$\varphi : G \longrightarrow \mathrm{GL}(V), \qquad n \longmapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

(a) (2 points) Let $U = \mathrm{span}_{\mathbb{Q}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \subseteq V$. Show that U is an $\mathbb{Q}[G]$–submodule of V.

A $\mathbb{Q}[G]$–submodule of V is precisely a $\mathbb{Q}$–vector subspace of V that is invariant under the action of G, so to prove the claim it suffices to check that the subspace U is invariant under the action of G.
Given an arbitrary element $u = \begin{bmatrix} a \\ 0 \end{bmatrix}$ in U (with $a \in \mathbb{Q}$), and an arbitrary group element $n \in \mathbb{Z}$, we have

$$\Big(\phi(n)\Big)(u) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} \in U \qquad \text{as desired.}$$

Thus U is a $\mathbb{Q}[G]$–submodule of V.

(b) (4 points) Show that the short exact sequence of $\mathbb{Q}[G]$–modules

$$0 \longrightarrow U \hookrightarrow V \twoheadrightarrow V/U \longrightarrow 0 \qquad \text{does } \textbf{not} \text{ split.}$$

By definition, the sequence splits if we can find a direct complement U' to U in V. In other words, it splits if there is a $\mathbb{Q}[G]$–submodule U' in V such that $V \cong U \oplus U'$. To prove that it does not split, we will show that there is no G–invariant complement to U.

Because any splitting $V \cong U \oplus U'$ gives a splitting of the underlying $\mathbb{Q}$–vector space V into linear subspaces, the complement U' would have to be a 1–dimensional $\mathbb{Q}$–vector subspace of $\mathbb{Q}^2$ distinct from U. So suppose there is some 1–dimensional G–invariant subspace $U' = \mathrm{span}_{\mathbb{Q}}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$ for some $a, b \in \mathbb{Q}$. Because it is G–invariant, the action of G must map the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ to another vector in its span:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = c \begin{bmatrix} a \\ b \end{bmatrix} \qquad \text{for some } c \in \mathbb{Q}$$
$$\begin{bmatrix} a+b \\ b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix}$$

From the equation $b = cb$, either $c = 1$ or $b = 0$. If $c = 1$, then the equation $a + b = ca$ also implies that $b = 0$.

This shows that $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$, so $U' = U$ is the only 1-dimensional submodule of V.

(More generally, the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ must be an eigenvector of the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, so you can replace the above computation with your favourite method for finding eigenvectors, and show that U is the only 1–dimensional eigenspace.)

Hence U does not have any direct complement in V. The short exact sequence does not split.

(c) (1 point) Define an *equivalence* (or *isomorphism*) of G–representations

$$\varphi : G \to \mathrm{GL}(W) \qquad \text{and} \qquad \psi : G \to \mathrm{GL}(W')$$

An *equivalence* T of G–representations $\varphi : G \to \mathrm{GL}(W)$ and $\psi : G \to \mathrm{GL}(W')$ is a G–equivariant isomorphism of vector spaces $T : W \to W'$. In other words, it is a linear isomorphism T of vector spaces that commutes with the action of G in the sense that

$$T\Big(\Big(\varphi(g)\Big)(w)\Big) = \Big(\psi(g)\Big)(T(w)) \qquad \text{for all } g \in G \text{ and } w \in W.$$

Equivalently, $$T \circ \Big(\varphi(g)\Big) \circ T^{-1} = \psi(g) \qquad \text{for all } g \in G.$$

(d) (2 points) Show that the G–representations

$$\varphi : n \longmapsto \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \psi : n \longmapsto \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$$

are equivalent by finding an explicit isomorphism.

We seek an invertible 2×2 matrix T such that $T \circ \Big(\varphi(n)\Big) = \Big(\psi(n)\Big) \circ T$ for all n. It is possible to expand out the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and the invertibility condtition $ad - bc \neq 0$ to solve for the matrix entries a, b, c, d of T. We find that any matrix of the form $T = \begin{bmatrix} 0 & b \\ b & d \end{bmatrix}$ with $b, d \in \mathbb{Q}$, $b \neq 0$, would give a G-equivariant isomorphism.

It is simpler, however, to view T as a change of basis matrix,

$$T \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} T^{-1} = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$$

and observe that $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$ represent the same linear map with the roles of the two standard basis elements reversed. So we want the change-of-basis matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ that swaps the two standard basis elements. Observe:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$$

so $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is one such isomorphism.

3. Let R be a ring and D and R–module, and consider the functor $\mathrm{Hom}_R(D, -)$. Recall that we proved that this functor is left exact.

   (a) (2 points) Let $\phi : M \to N$ be a map of R–modules. Explain what it would mean for the induced map ϕ_* to be surjective.

   The map ϕ_* is surjective if every map of R–modules $f : D \to N$ can be expressed in the form $f = \phi_*(g) = \phi \circ g$ for some R–module map $g : D \to M$. In other words, given any map $f : D \to N$ we can find a map g (called a *lift* of f to M) that makes the diagram commute:

$$\begin{tikzcd} & D \arrow[dl, dashed, "\exists g"'] \arrow[d, "f"] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

   (b) (4 points) Suppose that D is a **free** R–module. Prove that if $\phi : M \to N$ is a surjective map, then ϕ_* is a surjective map.

   (This proves that $\mathrm{Hom}_R(D, -)$ is exact, that is, free R–modules are *projective*.)

   Suppose that D is a free module with basis A. Consider a surjective map $\phi : M \to N$ and any map $f : D \to N$. Since ϕ surjects, there exists preimages under ϕ for each element in the set $\{f(a) \in N \mid a \in A\}$. Let $\tilde{a}$ be a choice of preimage of $f(a)$ in M. Then we can define a map of sets

$$\begin{aligned} A &\longrightarrow M \\ a &\longmapsto \tilde{a} \in \phi^{-1}(f(a)) \end{aligned}$$

   By the universal property of the free module, the map $A \to M$ extends to a map of R–modules $g : D \to M$ satisfying $g(a) = \tilde{a}$:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & D \arrow[dl, dashed, "\exists! g"'] \arrow[d, "f"] \\ M \arrow[r, "\phi"] & N \end{tikzcd}$$

   Moreover, notice that $\quad \phi \circ g(a) = \phi(\tilde{a}) = f(a) \quad$ by construction,

   so the maps $\phi \circ g$ and f are equal when restricted to the basis A. Hence $\phi \circ g$ and f must be equal as maps on $D \to N$. (This may be viewed as a consequence of 'uniqueness' in the universal property of the free module on A, or otherwise as the general result that a map of R–modules is completely determined by its values on a generating set.)

   Thus we have constructed a lift g for $f = \phi \circ g = \phi_*(g)$.