1. Let M be a right R–module, and N a left R–module.
   (a) Describe an explicit construction of the tensor product $M \otimes_R N$ as a quotient of abelian groups.
   (b) State the universal property of the tensor product.
   (c) Verify that the explicit construction satisfies the universal property.
2. Let M be a right R–module, N a left R–module, and L an abelian group. Classify all functions $M \times N \to L$ that are both R–balanced and maps of abelian groups.
3. Let R and S be rings. Verify that the abelian group $R \otimes_{\mathbb{Z}} S$ has a ring structure with multiplication defined by $(r_1 \otimes s_1)(r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 s_2)$.
4. Let S and R be rings. Define an (R, S)–*bimodule*, and prove that an (R, S)–bimodule structure on an abelian group M is equivalent to a left module structure over the ring $R \otimes_{\mathbb{Z}} S^{\text{op}}$.
5. Define *extension of scalars* to a ring R from a subring S. Show by example that an S–module M may embed into the R–module obtained by extension of scalars, and it may not embed.
6. (a) Suppose that S is a subring of R. Prove that if F is a free S–module on basis A, then $R \otimes_S F$ is a free R–module on basis $\{1 \otimes a \mid a \in A\} \cong A$.
   (b) Conclude that if V is an n-dimensional real vector space on basis $e_1, \ldots, e_n$, then $\mathbb{C} \otimes_{\mathbb{R}} V$ is an n-dimensional complex vector space with basis $1 \otimes e_1, \ldots, 1 \otimes e_n$.
7. What is the complex dimension of the vector spaces $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^s \otimes_{\mathbb{R}} \mathbb{R}^t$ and $\mathbb{C}^t \otimes_{\mathbb{R}} \mathbb{R}^s$?
8. Prove that any element of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^3$ can be written as the sum of at most two simple tensors (Recall: a *simple* or *pure* tensor in $V \otimes_R W$ is an element of form $v \otimes w$).
9. Let V be a $\mathbb{C}[x]$–module where x acts by a linear transformation A, and let W be a $\mathbb{C}[x]$–module where x acts by a linear transformation B. If V and W have positive dimensions m and n over $\mathbb{C}$, is it possible that $V \otimes_{\mathbb{C}[x]} W$ could be zero? Is it possible that it could be mn-dimensional? Under what conditions could it be less than nm–dimensional?
10. Compute $(\mathbb{Z}/15\mathbb{Z} \oplus \mathbb{R}) \otimes_{\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Q}) \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$.
11. Prove or disprove: Suppose S is a subring of the commutative ring R, and M and N are R–modules. Then the tensor product $M \otimes_R N$ is a quotient of the tensor product $M \otimes_S N$.
12. Let R be an integral domain and M an R–module. Suppose that $x_1, \ldots, x_n$ is a maximal list of linearly independent elements. Prove that $Rx_1 + Rx_2 + \cdots + Rx_n$ is isomorphic to R^n, and that $M/(Rx_1 + Rx_2 + \cdots + Rx_n)$ is a torsion R–module.
13. Let R be an integral domain.
    (a) Suppose that A and B are R–modules of ranks a and b, respectively. Prove that $A \oplus B$ is an R–module of rank $a + b$.
    (b) Let R be an integral domain, and consider a short exact sequence of finite-rank R–modules:
    $$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$
    Show that rank(B) = rank(A)+ rank(C).
14. Let R be an integral domain, and I any **non-principal** ideal of R. Determine the rank of I, and prove that I is not a free R–module.
15. Find the lists of invariant factors and of elementary divisors for the finitely generated abelian group
    $$M \cong \mathbb{Z}^7 \oplus \frac{\mathbb{Z}}{20\mathbb{Z}} \oplus \frac{\mathbb{Z}}{18\mathbb{Z}} \oplus \frac{\mathbb{Z}}{75\mathbb{Z}}.$$

16. Let $R = M_{n\times n}(\mathbb{Q})$ be the ring of rational $n \times n$ matrices. Let $S \cong \mathbb{Q}$ be the subring of scalar matrices. Show that $\mathrm{End}_R(\mathbb{Q}^n) = S$ and $\mathrm{End}_S(\mathbb{Q}^n) = R$.

17. Suppose the following diagram is commutative and has exact rows. Prove that if m and p are injective, and l is surjective, then n is injective.

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\ \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} \\ A' & \xrightarrow[r]{} & B' & \xrightarrow[s]{} & C' & \xrightarrow[t]{} & D' \end{array}$$

18. Let k be a field, and x, y indeterminates. Prove or disprove the following isomorphism of k–modules: $k[x, y] \cong k[x] \otimes_k k[y]$.

19. Let k be a field and let V, W be k-vector spaces. Show that there is a natural isomorphism of k–modules:

$$\mathrm{Hom}_k(W, k) \otimes_k V \cong \mathrm{Hom}_k(W, V).$$

(By "natural isomorphism", I mean the map can be defined without choosing a basis for W or V.)

20. Let R be commutative and let M, N be R–modules. Show that there is a canonical isomorphism

$$M \otimes_R N \cong N \otimes_R M.$$

21. Let M, M_i be right R–modules and N, N_i be left R–modules. Use the universal property of the tensor product and the universal property of the direct sum to prove the following isomorphisms of abelian groups:

$$(M_1 \oplus M_2) \otimes_R N \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N) \qquad M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2)$$

22. Let V and W be vector spaces over a field $\mathbb{F}$ with bases $\{e_1, \dots, e_n\}$ and $\{f_1, \dots, f_m\}$, respectively.
    (a) Show that $\{e_i \otimes f_j\}_{i=1,j=1}^{n,m}$ is a basis for $V \otimes_{\mathbb{F}} W$.
    (b) It follows from part (a) that any element α of $V \otimes_{\mathbb{F}} W$ can be written in the form $\alpha = \sum_{i,j} c_{i,j}(e_i \otimes f_j)$. Prove that α can be expressed as a simple tensor (that is, in the form $v \otimes w$ for $v \in V, w \in W$) if and only if the matrix $(c_{i,j})$ has rank 1.

23. Classify (up to conjugacy) all linear maps $T : \mathbb{Q}^5 \to \mathbb{Q}^5$ with characteristic polynomial $c(x) = x^2(x-2)^3$.

24. Let M be a finitely generated module over a PID R. Give necessary and sufficient conditions on the elementary divisors of M for M to *irreducible.*

25. Let M be a simple R–module. Prove that M is cyclic. If M is cyclic, must M be simple?

26. Let V be a finite dimensional complex vector space and $T : V \to V$ a linear map. Under what conditions is the associated $\mathbb{C}[x]$–module V completely reducible?

27. Prove that 3×3 matrices over a field k are similar if and only if they have the same minimal and characteristic polynomials. Is this true of 4×4 matrices?

28. Prove that any square matrix A is similar to its transpose A^T.

29. Determine the rational and Jordan canonical form of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use these results to compute its characteristic and minimal polynomials, invariant factors, elementary divisors, eigenvalues, and dimensions of its (generalized) eigenspaces.

30. Determine representatives for all the conjugacy classes of $GL_2(\mathbb{F}_3)$.

31. Let k be a field and V a vector space over k. Prove that any group representation $G \to GL(V)$ extends uniquely to a map of rings $k[G] \to \operatorname{End}(V)$. Explain how this defines a $k[G]$–module structure on V.

32. Prove that there is a bijective correspondence between degree-1 representations of a group G, and degree-1 representations of its abelianization $G/[G, G]$.

33. Let G be a finite group, and $\mathbb{F}$ a field containing $\frac{1}{|G|}$.
   (a) State Maschke's theorem.
   (b) Show by example that if $|G|$ divides the characteristic of $\mathbb{F}$, then not all G–representations over $\mathbb{F}$ are completely reducible.

34. Let $\mathbb{F}$ be a field, G a finite group with order $|G|$ invertible in $\mathbb{F}$. Show that Maschke's theorem implies that every short exact sequence of $\mathbb{F}[G]$–modules splits.

35. Prove that isomorphic G–representations have the same character.

36. Let V be a G-representation. Show that the action of a group element $g \in G$ on V is G–equivariant if and only if g is in the center of G.

37. Prove that if U is a complex irreducible representation of G, and $V = U \oplus U$, then there are infinitely many ways that V can be decomposed into two copies of U. What is $\operatorname{Hom}_{\mathbb{C}[G]}(U, V)$? $\operatorname{Hom}_{\mathbb{C}[G]}(V, U)$?

38. Let V be an irreducible complex representation of a finite group G. Show that the multiplicity of V in a G–representation U is equal to $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}[G]}(V, U) = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathbb{C}[G]}(U, V)$.

39. (a) Let $\mathbb{F}$ be a field. Given any finite set $B = \{b_1, \ldots, b_m\}$, with an action of G, show how to construct a permutation representation by G on the vector space over $\mathbb{F}$ with basis B. Show that each G-orbit of B corresponds to a G subrepresentation of V.
   (b) Suppose that G acts transitively on the basis B (more generally, you can reply this result to the span of each G-orbit of B). Show that the diagonal subspace $D = \langle b_1 + b_2 + \cdots + b_m \rangle$ is invariant under G, and that G acts on it trivially. Show the orthogonal complement of D,

$$D^{\perp} = \left\{ a_1 b_1 + \ldots + a_m b_m \;\middle|\; \sum a_i = 0 \right\}$$

   is also invariant under the action of G, so that V decomposes as a direct sum of G subrepresentations $V \cong D \oplus D^{\perp}$. Compute the degrees of D and $D^{\perp}$.
   (c) Suppose that G acts transitively on the basis B. Prove that $D^{\perp}$ does not contain any vectors fixed by G (and therefore does not contain any trivial subrepresentations).
   (d) Show that the regular representation $V \cong \mathbb{F}[G]$ decomposes into a direct sum of invariant subspaces:

$$\left\{ \sum_{g \in G} a e_g \;\middle|\; a \in \mathbb{F} \right\} \bigoplus \left\{ \sum_{g \in G} a_g e_g \;\middle|\; \sum_{g \in G} a_g = 0 \right\}$$

(e) Use this decomposition and the averaging map to give a new proof that the multiplicity of the trivial representation in $\mathbb{F}[G]$ is 1.

40. Let V be a vector space over $\mathbb{F}$ with basis $x_1, \ldots, x_n$. Construct an isomorphism of rings $\mathrm{Sym}^* V \cong \mathbb{F}[x_1, x_2, \ldots, x_n]$ that commutes with scalar multiplication by $\mathbb{F}$.

41. Let M be a module over a commutative ring R. Show that the constructions T^*M, Sym^*M, and $\wedge^* M$ define functors from R–modules to rings (in fact, R–algebras).

42. Prove that a finite group G is abelian if and only if all its complex irreducible representations are 1-dimensional.

43. (a) Let g be a diagonalizable linear transformation acting on a vector space V, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Describe the set of eigenvalues of the map induced by g on the spaces $V \otimes V$, $V \otimes V \otimes V$, $\wedge^2 V$, $\wedge^3 V$, $\mathrm{Sym}^2 V$, and $\mathrm{Sym}^3 V$.

    (b) Suppose G is a finite group, and V a representation of G. Derive formulas for the characters of the representations $V \otimes V$, $V \otimes V \otimes V$, $\wedge^2 V$, $\wedge^3 V$, $\mathrm{Sym}^2 V$, and $\mathrm{Sym}^3 V$ in terms of the character χ_V for V.

44. Let G be a group. When are two 1-dimensional representations of G isomorphic?

45. Let A be a finite **abelian** group.

    (a) Explain why the complex representations of A are precisely the set of group homomorphisms from A to the multiplicative group of units $\mathbb{C}^\times$ of $\mathbb{C}$.

    (b) Let $a \in A$ be an order of element k. What are the possible homomorphic images of a in $\mathbb{C}^\times$?

    (c) Let A be a finite cyclic group of order n. State the number of non-isomorphic representations of A, and describe these explicitly.

    (d) Let ξ_n denote an n^{th} root of unity. Write down the character tables for the groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.