1. Let M be an R–module.
   (a) Prove that its annihilator $\operatorname{ann}(M)$ is a two-sided ideal of R, and that there is a well-defined action of $R/\operatorname{ann}(M)$ on M.
   (b) Prove that this action is faithful.

2. For each of the following, prove the statement or find a counterexample. Let M be an R–module, I a (right) ideal of R, and N a R–submodule.
   (a) If $\operatorname{ann}(N) = I$, then $\operatorname{ann}(I) = N$.
   (b) If $\operatorname{ann}(I) = N$, then $\operatorname{ann}(N) = I$.

3. (a) Let $\mathbb{F}$ be a field. Let V be an $\mathbb{F}[x]$ module where x acts by a linear transformation T. Under what conditions will a linear map S be an element of $\operatorname{End}_{\mathbb{F}[x]}(V)$?
   (b) Let $R = \mathbb{Z}[\sqrt{2}] = \{a + \sqrt{2} \mid a, b \in \mathbb{Z}\}$ and consider R as a module over itself. As an abelian group, $M = \mathbb{Z}[\sqrt{2}]$ is isomorphic to $\mathbb{Z}^2$ under the identification $a + b\sqrt{2}$ with $(a, b) \in \mathbb{Z}^2$. Show that $\operatorname{End}_{\mathbb{Z}}(M)$ is the matrix ring $M_{2\times 2}(\mathbb{Z})$, and identify the subset of matrices $\operatorname{End}_R(M) \subseteq M_{2\times 2}(\mathbb{Z})$ that commute with the action of R.

4. State and prove the first isomorphism theorem for R–modules.

5. Give an example of an integral domain R and two non-isomorphic finitely generated torsion R–modules with the same annihilators.

6. Let V be a $\mathbb{C}[x]$–module such that V is finite dimensional as a vector space over $\mathbb{C}$. Prove that V is a torsion module.

7. Let $\phi : M \to N$ be a homomorphism of R–modules. Let I be a right ideal of R. Let $\operatorname{ann}_M(I)$ denote the annihilator of I in M, and $\operatorname{ann}_N(I)$ the annihilator of I in N. Prove or find a counterexample: $\phi(\operatorname{ann}_M(I)) \subseteq \operatorname{ann}_N(I)$.

8. Let M and N be R–modules, and I an ideal of R contained in $\operatorname{ann}(M)$ and $\operatorname{ann}(N)$. Show that any map of R–modules $\phi : M \to N$ is also a map of (R/I)–modules. Conclude that $\operatorname{Hom}_R(M, N) = \operatorname{Hom}_{R/I}(M, N)$.

9. (a) If $a \in R$, prove that $Ra \cong R/\operatorname{ann}(a)$, where $\operatorname{ann}(a)$ denotes the annihilator of the left ideal generated by a.
   (b) Let M be an R–module. For $a, b \in M$, let $A = \{a, b\}$. Prove or disprove: $RA \cong R/I$, where I is the annihilator of the submodule generated by a and b.

10. Let G be a group. Give three definitions of a *representation* of G, and explain why they are equivalent.

11. Find a faithful representation of the circle group $T \cong \mathbb{R}/2\pi\mathbb{Z}$ into $\mathrm{GL}_2(\mathbb{R})$.

12. Let G be a finite group, and consider the rational *regular representation*, the group ring $\mathbb{Q}[G]$ as a module over itself. Prove that if $|G| > 1$ then the regular representation always contains a proper nonzero G-invariant subrepresentation.

13. Let G be a finite group. Prove that all degree-1 representations of G are in bijective correspondence with degree-1 representations of its abelianization G^{ab}.

14. Let G be a finite cyclic group. Find all 1-dimensional representations of G, and determine which are inequivalent.

15. Let $\phi : G \to \mathrm{GL}_n(\mathbb{F})$ be a representation of a group G. Show that composing with the determinant map gives a map $g \to \det(\phi(g))$ that is a degree-1 representation of G.

16. Let M be and R–module with submodules A and B. Prove that the map $A \times B \longrightarrow A + B$ is an isomorphism if and only if $A \cap B = \{0\}$.

17. Give an example of a finitely generated R–module M and a submodule that is not finitely generated.

18. Let S be the set of all sequences of integers $(a_1, a_2, a_3, \ldots)$ that are nonzero in only finitely many components (in other words, all functions $\mathbb{N} \to \mathbb{Z}$ with finite support). Verify that S is a ring (without identity) under componentwise addition and multiplication. Is S a finitely generated S–module?

19. A student makes the following claim: "Since $\mathbb{Z}/2\mathbb{Z}$ is a subring of $\mathbb{Z}/4\mathbb{Z}$, we can let $\mathbb{Z}/2\mathbb{Z}$ act by left multiplication to give $\mathbb{Z}/4\mathbb{Z}$ the structure of a $\mathbb{Z}/2\mathbb{Z}$–module. Then $\mathbb{Z}/4\mathbb{Z}$ is a $\mathbb{Z}/2\mathbb{Z}$–vector space with 4 elements, so it must be isomorphic as a vector space to $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$." Prove that $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$ are not even isomorphic as abelian groups, and find the flaw in this argument.

20. (a) Let $M_1, \ldots M_n$ be R–modules, and N_i a submodule of M_i for all i. Prove that
$$\frac{M_1 \times M_2 \times \cdots \times M_n}{N_1 \times N_2 \times \ldots \times N_n} \cong \left(\frac{M_1}{N_1}\right) \times \left(\frac{M_2}{N_2}\right) \times \cdots \times \left(\frac{M_n}{N_n}\right).$$
    (b) Let I be any left ideal of R, and let $IR^n = \{\text{finite sums } \sum a_i x_i \mid a_i \in I, x_i \in R^n\}$. Prove that
$$\frac{R^n}{IR^n} \cong \frac{R}{IR} \times \frac{R}{IR} \times \cdots \times \frac{R}{IR}.$$

21. A *central idempotent* e in a ring R is an central element satisfying $e^2 = e$.
    (a) What are the central idempotents in $\mathbb{Z}^n$?
    (b) What are the central idempotents in $\mathrm{M}_2(\mathbb{Q})$, the 2×2 rational matrices?
    (c) Show that if e is a central idempotent in R and M an R–module, then $M \cong eM \oplus (1-e)M$.

22. Suppose that R is a ring and that S is a subring.
    (a) Suppose that F is a free R–module. Prove or disprove: F is a free S–module after restriction of scalars to S.
    (b) Suppose that M is an R–module that is free as an S–module after restriction to S. Prove or disprove: M must be a free R–module.

23. Let R be a ring.
    (a) Give the definition of a *free* R–module on a set A.
    (b) Given a set A, explain how to construct a free R–module $F(A)$ on A.
    (c) State the universal property for a free R–module.
    (d) Verify that $F(A)$ satisfies this universal property.
    (e) Prove that the universal property determines $F(A)$ uniquely up to unique isomorphism.
    (f) Show that F defines a covariant functor from the category of sets to the category of R–modules.

24. Let R be a commutative ring, and let A, B, M be R–modules. Use the universal property of the direct sum to prove the isomorphisms of R–modules:
    (a) $\mathrm{Hom}_R(A \oplus B, M) \cong \mathrm{Hom}_R(A, M) \oplus \mathrm{Hom}_R(B, M)$
    (b) $\mathrm{Hom}_R(M, A \oplus B) \cong \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(M, B)$

25. Let R be a commutative ring. If M and N are free R–modules, will the R–module $\mathrm{Hom}_R(M, N)$ be free? If $\mathrm{Hom}_R(M, N)$ is a free R–module, must M and N be free?

26. Find two non-equivalent extensions of the abelian groups $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/6\mathbb{Z}$.

27. Prove that every short exact sequence of vector spaces splits.

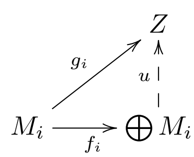
28. State the definition of a category, and the definition of a covariant functor.

29. Let $\mathcal{C}$ be a category. Prove that if $X \in \text{ob}(\mathcal{C})$, then the identity morphism id_X is unique. Further prove that if $f \in \text{Hom}_{\mathcal{C}}(X, Y)$, is an isomorphism, then its inverse f^{-1} is unique.

30. (a) Prove that in the category of sets, a map is monic iff it is injective, and epic iff it is surjective.
    (b) Prove that in any category the composition of monomorphisms (respectively, epimorphisms, or isomorphisms) is a monomorphisms (respectively, an epimorphism, or isomorphism).
    (c) Prove that isomorphisms are both monic and epic.

31. Prove or disprove the following statements.
    (a) If $f : A \to B$ is a monomorphism (respectively, epimorphism) in a category $\mathcal{C}$, then the image of f under any (covariant) functor $\mathcal{C} \to \mathcal{D}$ must be a monomorphism (respectively, epimorphism) in $\mathcal{D}$.
    (b) If $0 \to A \to B \to C \to 0$ is a short exact sequence in the category of R–modules, then its image under any (covariant) functor R–$\underline{\text{Mod}} \to R$–$\underline{\text{Mod}}$ must be an exact sequence.

32. **(Coproducts of families).** Prove that the direct sum of R–modules $\bigoplus_{i \in I} M_i$, along with the inclusions $f_i : M_i \to \bigoplus_{i \in I} M_i$, satisfies the following universal property: whenever there is a family of maps $\{g_i : M_i \to Z \mid i \in I\}$ there is a unique map u making the following diagrams commute:

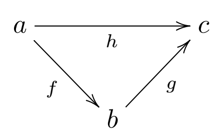


Explain why this universal property can be taken as the definition of the direct sum of R–modules.

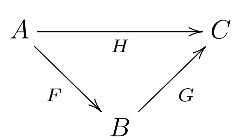
33. Let R be an integral domain. Prove or disprove: The map of R–modules that takes an R–module M to its R–submodule $\text{Tor}(M)$ and takes a map $f : M \to N$ to its restriction $f|_{\text{Tor}(M)}$ defines an exact covariant functor R–$\underline{\text{Mod}} \to R$–$\underline{\text{Mod}}$.

34. Let R be a ring. Define a functor on the category R–$\underline{\text{Mod}}$ that takes an R–module M to the R–module $M \oplus M$. Verify that your construction is functorial.

35. Consider the category J that consists of 3 objects and 3 morphisms (plus the identity morphisms, which are not shown):



with composition law $h = g \circ f$. A functor from J to a category $\mathcal{C}$ determines a choice of 3 (not necessarily distinct) objects A, B, C of $\mathcal{C}$ and morphisms F, G, H satisfying $H = G \circ F$.



Any such functor is called a *diagram of shape J in $\mathcal{C}$*. More generally, some mathematicians define the word *diagram* in a category $\mathcal{C}$ to mean a functor from a suitably chosen category J. Find a category J so that diagrams of shape J in $\mathcal{C}$ encode the following data:

(a) Any object in $\mathcal{C}$

(b) Any morphism $F : A \to B$ in $\mathcal{C}$

(c) Any isomorphism $F : A \to B$ in $\mathcal{C}$

(d) Any commuting square in $\mathcal{C}$

(e) An object in $\mathcal{C}$ with two commuting endomorphisms

36. Let R be a ring. Define the *forgetful functor* from R–modules to abelian groups, and show that it is an exact functor.

37. Recall that a R–module I is called *injective* if the (contravariant) functor $\mathrm{Hom}_R(-, I)$ is exact. Prove that $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$–module, but is not an injective $\mathbb{Z}$–module.

38. The rows of the following diagram are exact. Prove that if m and p are surjective and q is a injective, then n is surjective.

$$\begin{array}{ccccccc} B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B' & \xrightarrow[s]{} & C' & \xrightarrow[t]{} & D' & \xrightarrow[u]{} & E' \end{array}$$

39. Let R be a ring and M a fixed R–module. Verify that the following maps each define a functor of categories. Explain how to define the functor on morphisms, determine whether it is covariant or contravariant, and verify that the map is functorial.

(a)

$$\begin{aligned} \mathrm{Hom}_R(M, -) : R\text{–}\underline{\mathrm{Mod}} &\longrightarrow \underline{\mathrm{Ab}} \\ N &\longmapsto \mathrm{Hom}_R(M, N) \end{aligned}$$

(b)

$$\begin{aligned} \mathrm{Hom}_R(-, M) : R\text{–}\underline{\mathrm{Mod}} &\longrightarrow \underline{\mathrm{Ab}} \\ N &\longmapsto \mathrm{Hom}_R(N, M) \end{aligned}$$

40. Determine whether the functor $\mathrm{Hom}_R(M, -)$ from R–modules to abelian groups is left exact, right exact, both, or neither.