1. Let F be an R–module and $\iota : B \to F$ a map of sets. Show that if F (along with the data of the map ι) satisfies the universal property of the free R–module on B, then ι must be injective.

2. Let R be an integral domain, $a \in R$, and M an R–module. Describe the R–module $\mathrm{Hom}_R\left(\frac{R}{(a)}, M\right)$.

3. Let M be a right R–module, and N a left R–module.
   (a) Describe an explicit construction of the tensor product $M \otimes_R N$ as a quotient of abelian groups.
   (b) State the universal property of the tensor product.
   (c) Verify that the explicit construction satisfies the universal property.

4. Let M be a right R–module, N a left R–module, and L an abelian group. Classify all functions $M \times N \to L$ that are both R–balanced and maps of abelian groups.

5. Let M be an abelian group and R a ring. Show that an R–module structure on M defines a map of abelian groups $R \otimes_{\mathbb{Z}} M \to M$. Which maps $R \otimes_{\mathbb{Z}} M \to M$ arise in this way?

6. Let R and S be rings.
   (a) Verify that the abelian group $R \otimes_{\mathbb{Z}} S$ has a ring structure with multiplication defined by
   $$(r_1 \otimes s_1)(r_2 \otimes s_2) = (r_1 r_2) \otimes (s_1 s_2).$$
   (b) Define an (R, S)–*bimodule*, and prove that an (R, S)–bimodule structure on an abelian group M is equivalent to a left module structure over the ring $R \otimes_{\mathbb{Z}} S^{\mathrm{op}}$.

7. Let $S \subseteq R$ be a subring, and M an S–module.
   (a) Define the *extension of scalars* of M from S to R.
   (b) Let
   $$\begin{aligned} \iota : M &\longrightarrow R \otimes_S M \\ m &\longmapsto 1 \otimes m \end{aligned}$$
   Show that ι is a well-defined map of abelian groups, and moreover commutes with the action of S.
   (c) Let $\mathcal{F} : R\text{–}\underline{\mathrm{Mod}} \to S\text{–}\underline{\mathrm{Mod}}$ be the forgetful functor that only remembers the action of the subring S. Show that the R–module $R \otimes_S M$ is uniquely characterized by the following universal property: If L is an R–module, and $\varphi : M \to \mathcal{F}(L)$ a map of S–modules, then φ factors uniquely through the map ι to give a map of R–modules $\Phi : R \otimes_S M \to L$.
   (d) Conclude that there is a isomorphism of abelian groups
   $$\mathrm{Hom}_S(M, \mathcal{F}(L)) \cong \mathrm{Hom}_R(R \otimes_S M, L)$$
   Since this map is *natural*, the functors $\mathcal{F}$ and $R \otimes_S -$ are an adjoint pair.

8. Let $S \subseteq R$ be a subring, and M an S–module. Show by example that an S–module M may embed into the R–module obtained by extension of scalars, and it may not embed.

9. Show that if V is an n-dimensional real vector space on basis $e_1, \ldots, e_n$, then $\mathbb{C} \otimes_{\mathbb{R}} V$ is an n-dimensional complex vector space with basis $1 \otimes e_1, \ldots, 1 \otimes e_n$.

10. What is the complex dimension of the vector spaces $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^s \otimes_{\mathbb{R}} \mathbb{R}^t$ and $\mathbb{C}^t \otimes_{\mathbb{R}} \mathbb{R}^s$?

11. Prove that any element of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^3$ can be written as the sum of at most two simple tensors.

12. Let V be a $\mathbb{C}[x]$–module where x acts by a linear transformation A, and let W be a $\mathbb{C}[x]$–module where x acts by a linear transformation B. If V and W have positive dimensions m and n over $\mathbb{C}$, is it possible that $V \otimes_{\mathbb{C}[x]} W$ could be zero? Is it possible that it could be mn-dimensional? Give examples.

13. Compute $(\mathbb{Z}/15\mathbb{Z} \oplus \mathbb{R}) \otimes_{\mathbb{Z}} (\mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Q}) \otimes_{\mathbb{Z}} (\mathbb{Z}/3\mathbb{Z})$.

14. Prove that if $a \neq b \in \mathbb{Q}$, then $\dfrac{\mathbb{Q}[x,y]}{(x-a)} \otimes_{\mathbb{Q}[x,y]} \dfrac{\mathbb{Q}[x,y]}{(x-b)} \cong 0$.

15. Prove or disprove: Suppose S is a subring of the commutative ring R, and M and N are R–modules. Then the tensor product $M \otimes_R N$ is a quotient of the tensor product $M \otimes_S N$.

16. Let R be an integral domain and M an R–module. Suppose that $x_1, \ldots, x_n$ is a maximal list of linearly independent elements. Prove that $Rx_1 + Rx_2 + \cdots + Rx_n$ is isomorphic to R^n, and that $M/(Rx_1 + Rx_2 + \cdots + Rx_n)$ is a torsion R–module.

17. Let R be an integral domain. Define the *rank* of an R–module M to be the maximal cardinality of any list of linearly independent elements in M. Prove that the rank of a $\mathbb{Z}$–module M is equal to the dimension of $\mathbb{Q} \otimes_{\mathbb{Z}} M$.

18. Let R be an integral domain. Recall that the definition of *rank* from Question 17.
    (a) Suppose that A and B are R–modules of ranks a and b, respectively. Prove that $A \oplus B$ is an R–module of rank $a + b$.
    (b) Consider a short exact sequence of finite-rank R–modules:
    $$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\phi} C \longrightarrow 0$$
    Show that $\text{rank}(B) = \text{rank}(A) + \text{rank}(C)$.

19. Let R be an integral domain, and I any **non-principal** ideal of R. Determine the rank of I, and prove that I is not a free R–module.

20. Let $R = M_{n\times n}(\mathbb{Q})$ be the ring of rational $n \times n$ matrices. Let $S \cong \mathbb{Q}$ be the subring of scalar matrices. Show that $\text{End}_R(\mathbb{Q}^n) = S$ and $\text{End}_S(\mathbb{Q}^n) = R$.

21. Let k be a field, and x, y indeterminates. Prove or disprove the following isomorphism of k–modules: $k[x,y] \cong k[x] \otimes_k k[y]$.

22. Let R be commutative and let M, N be R–modules. Show that there is a canonical isomorphism
$$M \otimes_R N \cong N \otimes_R M.$$

23. Let M, M_i be right R–modules and N, N_i be left R–modules. Use the universal property of the tensor product and the universal property of the direct sum to prove the following isomorphisms of abelian groups:
$$(M_1 \oplus M_2) \otimes_R N \cong (M_1 \otimes_R N) \oplus (M_2 \otimes_R N) \qquad M \otimes_R (N_1 \oplus N_2) \cong (M \otimes_R N_1) \oplus (M \otimes_R N_2)$$

24. Let V be a vector space over $\mathbb{F}$ with basis $x_1, \ldots, x_n$. Construct an isomorphism of $\mathbb{F}$–algebras
$$\text{Sym}^* V \cong \mathbb{F}[x_1, x_2, \ldots, x_n]$$
(ie, an isomorphism of rings that commutes with scalar multiplication by $\mathbb{F}$).

25. Let M be a simple R–module. Prove that M is cyclic. If M is cyclic, must M be simple?

26. Let V be a finite dimensional complex vector space and $T : V \to V$ a linear map. Under what conditions is the associated $\mathbb{C}[x]$–module V completely reducible?

27. Let G be a group. Give three definitions of a *representation* of G.

28. Let k be a field and V a vector space over k. Prove that any group representation $G \to GL(V)$ extends uniquely to a map of rings $k[G] \to \mathrm{End}(V)$. Explain how this defines a $k[G]$–module structure on V.

29. Let G be a group and V an $k[G]$–module. Explain why any map of **sets** $f : G \to V$ extends uniquely to a map of k–modules $k[G] \to V$. Under what (necessary and sufficient) conditions will this be a map of G–representations?

30. Find a faithful representation of the circle group $T \cong \mathbb{R}/2\pi\mathbb{Z}$ into $\mathrm{GL}_2(\mathbb{R})$.

31. Let G be a finite group. Prove that all degree-1 representations of G are in bijective correspondence with degree-1 representations of its abelianization G^{ab}.

32. For any $n \geq 2$, define the S_n–representations $\underline{\text{Trv}}$ and $\underline{\text{Alt}}$. Prove that these are the only 1-dimensional S_n–representations.

33. Let G be a finite group, and $\mathbb{F}$ a field containing $\frac{1}{|G|}$.
    (a) State Maschke's theorem.
    (b) Show that Maschke's theorem implies that every short exact sequence of $\mathbb{F}[G]$–modules splits.
    (c) Show by example that if $|G|$ divides the characteristic of $\mathbb{F}$, then not all G–representations over $\mathbb{F}$ are completely reducible.

34. Prove that if U is a complex irreducible representation of G, and $V = U \oplus U$, then there are infinitely many ways that V can be decomposed into two copies of U. What is $\mathrm{Hom}_{\mathbb{C}[G]}(U, V)$? $\mathrm{Hom}_{\mathbb{C}[G]}(V, U)$?

35. Let V be an irreducible complex representation of a finite group G. Show that the multiplicity of V in a G–representation U is equal to $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(V, U) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(U, V)$.

36. (a) Let $\mathbb{F}$ be a field. Given any finite set $B = \{b_1, \ldots, b_m\}$, with an action of G, show how to construct a permutation representation by G on the vector space over $\mathbb{F}$ with basis B. Show that each G-orbit of B corresponds to a G subrepresentation of V.
    (b) Suppose that G acts transitively on the basis B (more generally, you can reply this result to the span of each G-orbit of B). Show that the diagonal subspace $D = \langle b_1 + b_2 + \cdots + b_m \rangle$ is invariant under G, and that G acts on it trivially. Show the orthogonal complement of D,

    $$D^{\perp} = \left\{ a_1 b_1 + \ldots + a_m b_m \;\middle|\; \sum a_i = 0 \right\}$$

    is also invariant under the action of G, so that V decomposes as a direct sum of G subrepresentations $V \cong D \oplus D^{\perp}$. Compute the degrees of D and $D^{\perp}$.
    (c) Suppose that G acts transitively on the basis B. Prove that $D^{\perp}$ does not contain any vectors fixed by G (and therefore does not contain any trivial subrepresentations).
    (d) Show that the regular representation $V \cong \mathbb{F}[G]$ decomposes into a direct sum of invariant subspaces:

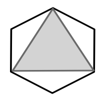
    $$\left\{ \sum_{g \in G} a e_g \;\middle|\; a \in \mathbb{F} \right\} \bigoplus \left\{ \sum_{g \in G} a_g e_g \;\middle|\; \sum_{g \in G} a_g = 0 \right\}$$

    (e) Use this decomposition and the averaging map to give a new proof that the multiplicity of the trivial representation in $\mathbb{F}[G]$ is 1.

37. Prove that a finite group G is abelian if and only if all its complex irreducible representations are 1-dimensional.

38. It is a nonobvious fact that all values of the irreducible complex characters of the symmetric groups are integer-valued. Prove that if V is an irreducible representation of S_n of degree at least 2, then there must be at least one conjugacy class of S_n where χ_V takes on the value zero.

39. (a) Use character theory to decompose the S_3–representation $\underline{\text{Alt}} \otimes_{\mathbb{C}} \mathbb{C}^3$.Verify your computation by finding an explicit basis for each irreducible constituent.
    (b) The symmetric group S_3 is the symmetry group of an equilateral triangle. If we inscribe the triangle inside a regular hexagon as shown,

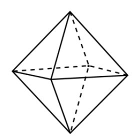


there is an induced action on the hexagon.

(i) In particular, there is an induced action on the set of vertices of the hexagon, and so an action on the free $\mathbb{C}$–module on this set. Compute the decomposition of the representation into irreducible components.
(ii) Do the same for the free $\mathbb{C}$–module on the set of edges of the hexagon.

Bonus: In both cases (i) and (ii), find explicit bases for the irreducible constituents.

40. Consider the complex S_4–representation $\mathbb{C}^4 \cong \underline{\text{Trv}} \oplus \underline{\text{Std}}$.
    (a) Prove that $\underline{\text{Std}}$ is irreducible.
    (b) Compute the character table of S_4.
    (c) Compute the characters of $\wedge^3\mathbb{C}^4$, and compute its decomposition into irreducible representations.
    (d) Compute the character of $\text{Hom}_{\mathbb{C}}(\underline{\text{Std}}, \underline{\text{Std}})$, and its decomposition into irreducible representations.
    (e) S_4 is the group of rigid motions of an octahedron (acting on the four pairs of opposite faces).



There is an induced action on the set of 6 vertices of the octahedron, and therefore on the free $\mathbb{C}$–module on this set. Compute the decomposition of this representation into irreducibles.

41. Let V and W be complex reprsentations of a finite group G.
    (a) Describe the G–representation structure on $\text{Hom}_{\mathbb{C}}(V, W)$.
    (b) Prove that $\text{Hom}_{\mathbb{C}}(V, W)^G \cong \text{Hom}_{\mathbb{C}[G]}(V, W)$.
    (c) Let ψ_{av} denote the averaging map, as applied to the representation $\text{Hom}_{\mathbb{C}}(V, W)$. Prove that this is a projection onto $\text{Hom}_{\mathbb{C}[G]}(V, W)$.
    (d) Suppose that V and W are non-isomorphic irreducible representations, and let $T \in \text{Hom}_{\mathbb{C}}(V, W)$. What is $\psi_{av}(T)$?
    (e) Now suppose that $V \cong W$ is irredcible. According to Schur's Lemma, $\psi_{av}(T)$ must be scalar multiplication. What is the scalar?

42. Let V be a nonzero representation of a finite group G. Show that $\text{Hom}_{\mathbb{C}}(V, V)^G$ is nonzero, and describe a basis of matrices in $\text{Hom}_{\mathbb{C}}(V, V)$ for this subrepresentation.

43. Let A be a finite **abelian** group.
    (a) Explain why the complex representations of A are precisely the set of group homomorphisms from A to the multiplicative group of units $\mathbb{C}^\times$ of $\mathbb{C}$.
    (b) Let $a \in A$ be an element of order k. What are the possible homomorphic images of a in $\mathbb{C}^\times$?
    (c) Find all 1-dimensional complex representations of A.

(d) Classify the irreducible complex representations of A up to isomorphism.

(e) Write down the character tables for the groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

44. Let G be a finite group and V a $\mathbb{C}[G]$–module.
    (a) Show that V is irreducible if and only if $\langle \chi_V, \chi_V \rangle_G = 1$.
    (b) Prove V is the sum of two non-isomorphic irreducible representations if and only if $\langle \chi_V, \chi_V \rangle_G = 2$.
    (c) What are the possibilities for the decomposition of V if $\langle \chi_V, \chi_V \rangle_G = 3$? $\langle \chi_V, \chi_V \rangle_G = 4$?

45. Let V be a complex G–representation. Show that $\chi_{V^*}(g) = \chi_V(g^{-1})$ for all $g \in G$.

46. Let V be a G–representation over a field $\mathbb{F}$. Show that V is irreducible if and only if V^* is irreducible.

47. Let V be a G–representation over a field $\mathbb{F}$. Show that $V \cong V^*$ as G–representations if and only if V has a nondegenerate G–invariant bilinear form.

48. Let G be a finite group and V a complex G–representation. Find a formula for the character of the G–representation $\bigwedge^3 V$ (in the spirit of our formula for $\chi_{\bigwedge^2 V}$).

49. Let G be a finite group. Prove that for any irreducible complex representation V of G, $\dim_{\mathbb{C}}(V) \leq \sqrt{|G|}$. For which G and V do we have equality?

50. Let G be a finite group, V an $\mathbb{F}$–vector space, and $\rho : G \to \mathrm{GL}(V)$ a G–representation. You proved on Homework #6 that if $\mathbb{F} = \mathbb{C}$, then $\rho(g)$ is diagonalizable for every $g \in G$.
    (a) Suppose $\mathbb{F}$ is a subfield of $\mathbb{C}$. Using extension of scalars of $\mathbb{F}$ to $\mathbb{C}$, what can you say about the eigenvalues and trace of $\rho(g)$?
    (b) Show by example that $\rho(g)$ can fail to be diagonalizable (even after extending to the algebraic closure $\overline{\mathbb{F}}$) when $\mathbb{F}$ has positive characteristic.