Reading: Dummit–Foote Ch 10.1.

Please review the Math 122 Course Information posted on our webpage:
`http://web.stanford.edu/~jchw/2017Math122`.

Summary of definitions and main results

Definitions we've covered: left R–module, right R–module, R–submodule, endomorphism, free R–module of rank n, annihilator of a submodule, annihilator of a (right) ideal.

Main results: Two equivalent definitions of an R–module; the submodule criterion, equivalence of vector spaces over a field $\mathbb{F}$ and $\mathbb{F}$–modules; equivalence of abelian groups and $\mathbb{Z}$–modules; if I annihilates an R–module M then M inherits a (R/I)–module structure; structure of an $\mathbb{F}[x]$–module for a field $\mathbb{F}$.

Warm-Up Questions

The "warm-up" questions do not need to be submitted (and won't be graded), however, you're encouraged to work out their solutions!

1. State the definition / axioms for a ring R (which we assume has unit 1).
2. In class we gave the definition of a left R–module. Formulate the definition of a *right R–module M*.
3. Let R be a ring with 1 and M a left R–module. Prove the following:
   (a) $0m = 0$ for all m in M.
   (b) $(-1)m = -m$ for all m in M.
   (c) If $r \in R$ has a left inverse, and $m \in M$, then $rm = 0$ only if $m = 0$.
4. Show that if R is a commutative ring, then a left R–module structure on an abelian group M also defines a right R–module on M and vice versa. Is this true for noncommutative rings R?
5. (**Restriction of scalars**). Let M be an R–module, and let S be any subring of R. Explain how the R–module structure on M also gives M the structure of an S–module. This operation is called *restriction of scalars* from R to the subring S.
6. Verify that the axioms for a vector space over a field $\mathbb{F}$ are equivalent to the axioms for an $\mathbb{F}$–module.
7. Verify that the axioms for an abelian group M are equivalent to the axioms for a $\mathbb{Z}$–module structure on M. How does an integer n act on $m \in M$?
8. Let $\mathbb{F}$ be a field, and x a formal variable. Prove that modules V over the polynomial ring $\mathbb{F}[x]$ are precisely $\mathbb{F}$–vector spaces V with a choice of linear map $T : V \to V$. In Assignment Problem 5 we will see that different maps T give different $\mathbb{F}[x]$–module structures on V.
9. Prove the *submodule criterion*: If M is a left R–module and N a subset of M, then N is a left R–submodule if and only if
$$(i) \quad N \neq \varnothing \qquad \text{and} \qquad (ii) \quad x + ry \in N \text{ for all } x, y \in N \text{ and all } r \in R.$$
10. Consider R as a module over itself. Prove that the R–submodules of the module R are precisely the left ideals I of R.
11. Let R^n be the free module of rank n over R. Prove that the following are submodules:
    (a) $I_1 \times I_2 \times \cdots \times I_n$, with I_i a left ideal of R.

   (b) The i^{th} direct summand R of R^n.
   (c) $\{(a_1, a_2, \ldots, a_n) \in R^n \mid a_1 + a_2 + \cdots + a_n = 0\}$.

12. Let M be a left R–module. Show that the intersection of a (nonempty) collection of submodules is a submodule.

13. (a) Let M be an R–module and N an R–submodule. Prove that the annihilator $\operatorname{ann}(N)$ is a 2-sided ideal of R.
    (b) Let M be an R–module and I a right ideal of R. Show that $\operatorname{ann}(I)$ is an R–submodule of M.
    (c) Compute the annihilator of the ideal $3\mathbb{Z} \subseteq \mathbb{Z}$ in the $\mathbb{Z}$–module $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$.

14. (a) For p prime, an *elementary abelian p-group* is an abelian group G where $pg = 0$ for all $g \in G$. Prove that an elementary abelian p-group is a $\mathbb{Z}/p\mathbb{Z}$–module, equivalently, an $\mathbb{F}_p$–vector space.
    (b) Conversely, show that any $\mathbb{Z}/p\mathbb{Z}$–module M must satisfy $pm = 0$ for all $m \in M$, in other words, the underlying abelian group M must be an elementary abelian p-group.

15. A student makes the following claim: "Since $\mathbb{Z}/2\mathbb{Z}$ is a subring of $\mathbb{Z}/4\mathbb{Z}$, we can let $\mathbb{Z}/2\mathbb{Z}$ act by left multiplication to give $\mathbb{Z}/4\mathbb{Z}$ the structure of a $\mathbb{Z}/2\mathbb{Z}$–module. Then $\mathbb{Z}/4\mathbb{Z}$ is a $\mathbb{Z}/2\mathbb{Z}$–vector space with 4 elements, so it must be isomorphic as a vector space to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$." Prove that $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ are not even isomorphic as abelian groups, and find the flaw in this argument.

16. Let M be an R–module, and consider $\operatorname{Tor}(M)$ as defined in Assignment Question 4.
    (a) Find $\operatorname{Tor}(\mathbb{Z}/7\mathbb{Z})$ if $\mathbb{Z}/7\mathbb{Z}$ is consider a module over (i) $\mathbb{Z}$, (ii) $\mathbb{Z}/7\mathbb{Z}$, or (iii) $\mathbb{Z}/21\mathbb{Z}$.
    (b) Show that if R has zero divisors, then ever nonzero R–module has nonzero torsion elements.

17. **(Group theory review)** State the structure theorem for finitely generated abelian groups.

18. **(Linear algebra review)**
    (a) Define the following terms (as they apply to finite dimensional vector spaces)
       - *vector space* over $\mathbb{F}$; *vector subspace*
       - *linear dependence* and *linear independence* of a set of vectors
       - *spanning set* of vectors for a vector subspace
       - *basis* and *dimension* of a vector subspace
       - the *direct sum* of vector subspaces
    (b) If you have not already seen proofs that
       - linearly independent sets of vectors in a finite dimensional vector space V can be extended to a basis, and
       - all bases for V have the same cardinality so $\dim(V)$ is well-defined

       then take a look at Dummit-Foote Chapter 11.1.
    (c) Let T be a linear transformation on a finite-dimensional $\mathbb{F}$-vector space V. Define an *eigenvector* of T and its associated *eigenvalue*. Find all eigenvectors and eigenvalues of the following matrices, over $\mathbb{R}$ and over $\mathbb{C}$.

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

    (d) If T has a basis of eigenvectors, then such as basis is called an *eigenbasis*. What can you say about the structure of a matrix with an eigenbasis, and why is this important? Which of the above four matrices have eigenbases over $\mathbb{R}$, or over $\mathbb{C}$?

Assignment Questions

The following questions should be handed in.

1. **(Group theory review)** Suppose $m, n \geq 2$ are integers.
   (a) Prove that there is an injective map of abelian groups $\mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ if and only if $m|n$.
   (b) Prove that if this map exists, it is unique up to pre-composing with an automorphism of $\mathbb{Z}/m\mathbb{Z}$. This means if $g, g' : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ are injective maps, then $g' = g \circ f$ for some $f : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$. Conclude in particular that the image of an injective map is a uniquely determined subset of $\mathbb{Z}/n\mathbb{Z}$.
   (c) $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ is an abelian group under pointwise addition of maps. Compute this group (in terms of the classification of finitely generated abelian groups).

2. Let M be an abelian group (with addition), and R a ring.
   (a) Define an *endomorphism* of M, and show that the set of endomorphisms $\text{End}(M)$ of M form a ring under composition and pointwise addition.
   (b) Prove that a left R–module structure on M is equivalent to the data of a homomorphisms of rings $R \to \text{End}(M)$. Use this result to formulate an alternative definition of a left R–module.
   (c) What should the analogous definition be for right R–modules?
   (d) We have another name for the kernel of the map $R \to \text{End}(M)$. What is it?

3. Let M be an R–module, and $\phi : S \to R$ a homomorphism of rings. Show how the map ϕ can be used to define an S–module structure on M. Explain why *restriction of scalars* is a special case of this construction. (Warm up Problem 5.)

4. An element m in an R–module M is called a *torsion element* if $rm = 0$ for some nonzero $r \in R$. The set of torsion elements is denoted

$$\text{Tor}(M) := \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

   Prove that if R is an integral domain, then $\text{Tor}(M)$ is submodule of M.
   Remark: For commutative rings R, some sources only define $\text{Tor}(M)$ with respect to elements $r \in R$ that are not zero divisors.

5. Let V be a module over the polynomial ring $\mathbb{F}[x]$. For each of the following, classify all submodules of V. You may give your answer as a list (without proof), but be sure you understand how you could justify your solutions!
   (a) $V = \mathbb{F}^2$, and x acts by scalar multiplication by 5.
   (b) $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^2$, and x acts by rotation by $\frac{\pi}{3}$.
   (c) $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^3$, and x acts by a diagonalizable matrix with eigenvalues $0, 1, 2$. (Recall: A matrix is *diagonalizable* iff it has a basis of eigenvectors, equivalently, iff it is conjugate to a diagonal matrix.)
   (d) $V = \mathbb{F}[x]$, and x acts by multiplication by x as usual.
   (e) $V = \mathbb{F}[x]/(x^3)$, and x acts by multiplication by x as usual.

6. **Bonus (Optional).** For the following problem, you may assume (without proof) the structure theorem for finitely generated abelian groups. Let's call a subgroup M of the free abelian group $\mathbb{Z}^n$ *splittable* if there is some (not necessarily unique) submodule $C \subseteq \mathbb{Z}^n$ so that $M \oplus C = \mathbb{Z}^n$. For example, $M = \text{span}\{(1,1)\} \subseteq \mathbb{Z}^2$ is splittable and has complement $C = \text{span}\{(1,0)\}$. In contrast $M = \text{span}\{(2,4)\} \subseteq \mathbb{Z}^2$ is not splittable since $M \oplus C$ can never contain $(1,2)$ for any $C \subseteq \mathbb{Z}^2$.
   (a) Prove that $M \subseteq \mathbb{Z}^n$ is splittable if and only if $\mathbb{Z}^n/M$ is torsion-free.
   (b) Prove or find a counterexample: If A and B are splittable subgroups of $\mathbb{Z}^n$, then $A \cap B$ is splittable.