Reading: Dummit–Foote Ch. 10.5 & pp 911-913.

Summary of definitions and main results

Definitions we've covered: monomorphism, epimorphism, isomorphism, covariant and contravariant functors, forgetful functor, free functor, dual space functor, functors $\mathrm{Hom}_R(D, -)$ and $\mathrm{Hom}_R(-, D)$, exact, exact sequence, short exact sequence, extension of C by A, presentation, split short exact sequence.

Main results: in the category R-mod monomorphisms are precisely the injections, free functor $F : \underline{\text{Set}} \to R\text{–}\underline{\text{Mod}}$ is a covariant functor, $\mathrm{Hom}_R(D, -)$ is a covariant functor, Short Five Lemma, Splitting Lemma.

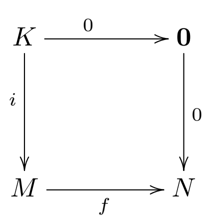
Warm-Up Questions

The "warm-up" questions do not need to be submitted (and won't be graded).

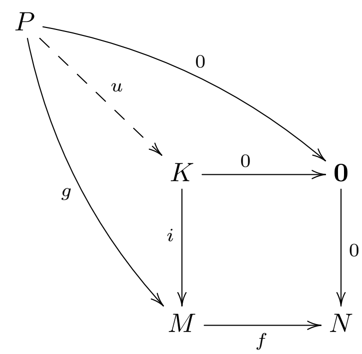
1. In this question, we will verify how the universal property defining free modules will fail for modules that are not free. Let $R = \mathbb{Z}$.
   (a) Consider the abelian group $\mathbb{Z}/m\mathbb{Z}$ and the subset $A = \{1 \pmod{m}\}$. Show that $\mathbb{Z}/m\mathbb{Z}$ fails to satisfy the universal property for being the free module on the basis A.
   Hint: Consider $M = \mathbb{Z}$ and any nonzero set map $A \to M$.
   (b) Consider the abelian group $\mathbb{Z}$ and the subset $A = \{2\}$. Show that $\mathbb{Z}$ fails to satisfy the universal property for being the free module on the basis A.
   Hint: Consider $M = \mathbb{Z}$ and the set map taking $2 \in A$ to $1 \in M$.
   (c) Consider the abelian group $\mathbb{Z} \oplus \mathbb{Z}$ and the subset $A = \{(1, 0)\}$. Show that $\mathbb{Z} \oplus \mathbb{Z}$ fails to satisfy the universal property for being the free module on the basis A.

2. (a) Prove that in the category of R–modules, a morphism is epic if and only if it is a surjective map.
   (b) Prove that in the category of rings, the map $\mathbb{Z} \to \mathbb{Q}$ is an epic morphism that is not surjective.

3. (a) A *zero object* $\mathbf{0}$ in a category is an object with the following property: For any object M, there is a unique morphism from M to $\mathbf{0}$, and a unique morphism from $\mathbf{0}$ to M. Show that if a category has a zero object, then it is unique up to unique isomorphism.
   (b) Let $\mathcal{C}$ be the category of R–modules, and show that the zero module $\{0\}$ is a zero object. This definition allows us to define the *zero map* 0 between R–modules M and N: it is the composition of the unique map $M \to \mathbf{0}$ with the unique map $\mathbf{0} \to N$.
   (c) Let $\mathcal{C}$ be the category R–modules. Verify that the kernel of an R–module map satisfies the following universal property. If $f : M \to N$ is a morphism in $\mathcal{C}$, then define the *kernel* $i : K \to M$ of f to be the map i such that $f \circ i$ is the zero morphism 0



   and satisfying the following: whenever there is a map of R–modules $g : P \to M$ such that $f \circ g = 0$, there is a unique map $u : P \to K$ such that $i \circ u = g$. In other words, there is a unique map u that makes the following diagram commute.



(d) Explain why this univeral property determines the map $i : K \to M$ up to unique isomorphism. Conclude that this universal property can be taken as the definition of the kernel of f.

4. Let $\mathscr{C}$ be a category containing objects A and B, and let F be a functor $F : \mathscr{C} \to \mathscr{D}$. Show that if A and B are isomorphic objects of $\mathscr{C}$, then $F(A)$ and $F(B)$ will be isomorphic objects of $\mathscr{D}$.

5. Given a group G, define a category $\mathscr{G}$ with a single object $\bigstar$ and morphisms $\mathrm{Hom}_{\mathscr{G}}(\bigstar, \bigstar) = \{g \mid g \in G\}$. The composition law is given by the group operation. Show that a function between groups $G \to H$ is a group homomorphism if and only if the corresponding map between categories $\mathscr{G} \to \mathscr{H}$ is a functor.

6. Let $\underline{\mathrm{f\,Set}}$ denote the category of finite sets and all functions between sets. Let $\mathscr{P} : \underline{\mathrm{f\,Set}} \to \underline{\mathrm{f\,Set}}$ be the function that takes a finite set A to its *power set* $\mathscr{P}(A)$, the set of all subsets of A. If $f : A \to B$ is a function of finite sets, let $\mathscr{P}(f) : \mathscr{P}(A) \to \mathscr{P}(B)$ be the function that takes a subset $U \subseteq A$ to the subset $f(U) \subseteq B$.
   (a) Show that $\mathscr{P}$ is a covariant functor.
   (b) What if we had instead defined $\mathscr{P}(f) : \mathscr{P}(B) \to \mathscr{P}(A)$ to take a subset $U \subseteq B$ to its preimage $f^{-1}(U) \subseteq A$ under f?

7. Let 0 denote the trivial abelian group. Give an example of a functor $F : \underline{\mathrm{Ab}} \to \underline{\mathrm{Ab}}$ such that $F(0) = 0$, and a functor $F : \underline{\mathrm{Ab}} \to \underline{\mathrm{Ab}}$ such that $F(0) \neq 0$.

8. Write down short exact sequences giving presentations of the following R–modules M. Give a list of generators and relations for M.
   (a) R^n
   (b) $R = \mathbb{Z}$, $M = \mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$
   (c) $R = \mathbb{Q}$, $M = \mathbb{Q}[x]/\langle x^2 + 1\rangle$
   (d) $R = \mathbb{C}[x, y]$, $M = \langle x, y\rangle$

9. (a) Find two non-isomorphic extensions of $\mathbb{Z}$–modules $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}$.
   (b) Find two non-isomorphic extensions of $\mathbb{Z}$–modules $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$.
   (c) How many extensions of $\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$ can you find?
   (d) Show that if $0 \to U \to W \to V \to 0$ is a short exact sequence of vector spaces, then $W \cong V \oplus U$.

10. Use the Splitting Lemma to show that if m and n are coprime, the following short exact sequence splits:

$$0 \longrightarrow \mathbb{Z}/m\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/mn\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0.$$

What if m and n are not coprime?

Assignment Questions

1. We proved in class that the map $\mathrm{Hom}_R(D, -) : R\text{–}\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}}$ is a covariant functor.
   (a) We have another name for the functor of abelian groups $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, -)$. What is it?
   (b) To which groups does the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ map the $\mathbb{Z}$–modules $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $(\mathbb{Z}/n\mathbb{Z})^p$, $\mathbb{Z}/n^p\mathbb{Z}$, and $\mathbb{Z}/m\mathbb{Z}$ (for m, n coprime)? Express your answers in terms of the classification of finitely generated abelian groups. (Since you already proved this result on Homework #1, you may state your solution without proof).
   (c) Describe the sequence of abelian groups and the maps obtained by applying $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$ to the following short exact sequences:

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

   In particular, state which resulting sequence is exact.

2. **(Short Five Lemma).** Consider a homomorphism of short exact sequences of R–modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\psi} & B & \xrightarrow{\varphi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow[\psi']{} & B' & \xrightarrow[\varphi']{} & C' & \longrightarrow & 0 \end{array}$$

   Prove the remaining step in the Short Five Lemma: If α and γ both surject, then β must also surject.

3. **(The Splitting Lemma).** Let R be a ring, and consider the short exact sequence of R–modules:

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \longrightarrow 0.$$

   Prove that the following are equivalent.
   (i) The sequence *splits*, that is, B is isomorphic to $A \oplus C$ such that ψ corresponds to the natural inclusion of A, and φ corresponds to the natural projection onto C.
   (ii) There is a map $\varphi' : C \to B$ such that $\varphi \circ \varphi'$ is the identity on C.

$$0 \longrightarrow A \xrightarrow{\psi} B \underset{\varphi'}{\overset{\varphi}{\rightleftarrows}} C \to 0$$

   (iii) There is a map $\psi' : B \to A$ such that $\psi' \circ \psi$ is the identity on A.

$$0 \to A \underset{\psi'}{\overset{\psi}{\rightleftarrows}} B \xrightarrow{\varphi} C \longrightarrow 0$$

   The maps φ' and ψ' are called *splitting homomorphisms.*

4. Define a ring R to be *(left) Noetherian* if R is Noetherian as a left module over itself, that is, every R–submodule of R is finitely generated. In this question we will show this definition is equivalent to the following alternate definition of a Noetherian ring: R is *(left) Noetherian* if **every** finitely generated left R–module is Noetherian.

(a) Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R–modules. Show that if A and C are finitely generated R–modules, then B is finitely generated.

(b) Suppose R is Noetherian as a left R–module. Let M be a submodule of R^n. Consider the short exact sequence of R–modules

$$0 \longrightarrow \{0\} \times R^{n-1} \longrightarrow R^n \xrightarrow{\pi_1} R \longrightarrow 0,$$

(Here, π_1 is the projection onto the first factor of R^n.) Show that we obtain a short exact sequence

$$0 \longrightarrow M \cap \left(\{0\} \times R^{n-1}\right) \longrightarrow M \longrightarrow \pi_1(M) \longrightarrow 0.$$

(c) Using parts (a) and (b) and induction on n, prove that R^n is a Noetherian R–module.

(d) Prove that an R–module N is finitely generated if and only if it is quotient of a finite rank free R–module R^n.

(e) Prove that a quotient of a Noetherian R–module is Noetherian.

(f) Conclude that any finitely generated R–module is Noetherian.

5. **Bonus (Optional).** Let N and M_i be R–modules for i in an index set I.

(a) Prove the following isomorphisms of abelian groups:

$$\text{(i)} \quad \mathrm{Hom}_R\left(N, \prod_{i \in I} M_i\right) \cong \prod_{i \in I} \mathrm{Hom}_R(N, M_i) \quad \text{(ii)} \quad \mathrm{Hom}_R\left(\bigoplus_{i \in I} M_i,\ N\right) \cong \prod_{i \in I} \mathrm{Hom}_R(M_i, N)$$

(b) Use these identifications to define (and verify) universal properties for the direct sum and direct product.