Reading: Dummit–Foote Ch 10.4.

Summary of definitions and main results

Definitions we've covered: tensor product of modules (as an abelian group), (S, R)–bimodule, tensor product of modules (as an S–module), R–balanced map, R–bilinear map, universal property of the tensor product (version for abelian groups, and version for R–module structure when R is commutative), tensor product of maps, extension of scalars.

Main results: Explicit construction of the tensor product $M \otimes_R N$, verification that the construction satisfies the universal property, $R/I \otimes_R N \cong N/IN$, tensor product distributes over direct sums, $R^n \otimes_R N \cong N^n$, tensor product is associative, using the universal property to compute specific tensor products.

Warm-Up Questions

1. Let R be a ring, let A be a right R–module and B a left R–module. Prove that the universal property of the tensor product defines the abelian group $A \otimes_R B$ uniquely up to unique isomorphism.
2. Let R be a ring with right R–module M and left R–module N. What is the additive identity in the tensor product $M \otimes_R N$? Explain why the elements $m \otimes 0$ and $0 \otimes n$ are zero for any $m \in M$ or $n \in N$.
3. (a) Explain why, when R is commutative, a left R–module M will also be a right R–module under the action $mr = rm$, and conversely any right R–module N has an induced left R–module structure.
   (b) Verify that these actions give M the structure of an (R, R)–bimodule (simply called an R–*bimodule*).
   (c) Why will these constructions generally not work when R is non-commutative?
4. Let R be a ring with right R–module M and left R–module N. Show that the natural map
$$M \times N \longrightarrow M \otimes_R N$$
is **not** a group homomorphism in general. What are the constraints on this map, as imposed by the defining relations of $M \otimes_R N$?
5. Let R and S be rings (possibly the same ring). Let M be a right R–module and N a left R–module. When will the tensor product $M \otimes_R N$ have the structure of an abelian group, and under what conditions will it additionally have the structure of an S–module?
6. Let R be a commutative ring. Let e_1, e_2, e_3 be a basis for the R^3 and let f_1, f_2, f_3, f_4 be a basis for R^4. Expand the tensor
$$(a_1e_1 + a_2e_2 + a_3e_3) \otimes (b_1f_1 + b_2f_2 + b_3f_3 + b_4f_4) \qquad \in R^3 \otimes_R R^4.$$
7. Let R be a ring with right R–module M and left R–module N. Which of the following maps are R–balanced? Which are homomorphisms of abelian groups? For the maps that are R–balanced, describe how they factor through the tensor product.
   (a) The identity map $M \times N \longrightarrow M \times N$.
   (b) The natural projections of $M \times N$ onto M and N.
   (c) The natural map $M \times N \longrightarrow M \otimes_R N$.
   (d) Suppose M and N are ideals of R. The multiplication map
$$\begin{aligned} M \times N &\longrightarrow R \\ (m, n) &\longmapsto mn \end{aligned}$$

(e) Suppose R is commutative. The matrix multiplication map

$$\begin{aligned} M_{n\times k}(R) \times M_{k\times m}(R) &\longrightarrow M_{n\times m}(R) \\ (A,B) &\longmapsto AB \end{aligned}$$

(f) Suppose R is commutative and M, N, P are R–modules. The composition map:

$$\begin{aligned} \mathrm{Hom}_R(M,N) \times \mathrm{Hom}_R(N,P) &\longrightarrow \mathrm{Hom}_R(M,P) \\ (f,g) &\longmapsto g \circ f \end{aligned}$$

(g) Suppose R is commutative. The dot product map:

$$\begin{aligned} R^n \times R^n &\longrightarrow R \\ (v,w) &\longmapsto v \cdot w \end{aligned}$$

(h) Suppose R is commutative. The cross product map:

$$\begin{aligned} R^3 \times R^3 &\longrightarrow R^3 \\ (v,w) &\longmapsto v \times w \end{aligned}$$

(i) Suppose R is commutative. The determinant map:

$$\begin{aligned} R^2 \times R^2 &\longrightarrow R \\ (v,w) &\longmapsto \det \begin{bmatrix} | & | \\ v & w \\ | & | \end{bmatrix} \end{aligned}$$

8. Fill in the details of the computations from class:

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \qquad \text{and} \qquad R/I \otimes_R N \cong N/IN$$

by verifying that the maps we constructed are R–balanced, R–bilinear, group homomorphisms, or mutual inverses, as appropriate.

9. Use the universal property of the tensor product to verify that $3 \otimes 6 \in \mathbb{Z}/12\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/20\mathbb{Z}$ is nonzero.

10. (a) Suppose that A is a finite abelian group. Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$.
    (b) Suppose that B is a finitely-generated abelian group. Show that $\mathbb{Q} \otimes_{\mathbb{Z}} B$ is a $\mathbb{Q}$–vector space. What determines its dimension?

11. Let R be a ring with right R–module M and left R–module N.
    (a) Show that there are always maps of abelian groups $N \to M \otimes_R N$.
    (b) Give an example where this map is injective, and an example where this map is not injective.

12. Let $V \cong \mathbb{C}^2$ be a complex vector space, and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix with respect to the standard basis e_1, e_2. Write down the matrix for the linear map induced by A on the four-dimensional vector space $V \otimes V$ with respect to the basis $e_1 \otimes e_1,\ e_1 \otimes e_2,\ e_2 \otimes e_1,\ e_2 \otimes e_2$.

13. Let V be a complex vector space. Let $T : V \to V$ be a diagonalizable linear map with eigenbasis $v_1, v_2, \ldots v_n$, and associated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. What are the eigenvalues of the map induced by T on $V \otimes V$, and what are the associated eigenvectors?

14. Let R be a ring and S a subring.
    (a) Give an example of R, S and an S–module that embeds into an R–module.
    (b) Give an example of R, S, and an S–module that cannot embed into any R–module.

Assignment Questions

1. In the following question, to 'compute' a finitely generated abelian group means to classify the group in terms of the structure theorem for finitely generated abelian groups, that is, to realize the group as a product of cyclic groups. To 'compute' a vector space means to determine its dimension.
   (a) For integers $m, n > 1$, compute the abelian groups $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Compare this to the group $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$.
   (b) For any integer $n > 1$, compute the abelian group $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$. Compute the group $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ and compare the two solutions.
   (c) Compute the rational vector space $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$.
   (d) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are **not** isomorphic as vector spaces over $\mathbb{R}$.
   (e) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ **are** isomorphic as vector spaces over $\mathbb{Q}$.

2. Let R be a commutative ring with ideals I and J. Prove the isomorphism of R–modules:

$$R/I \otimes_R R/J \longrightarrow R/(I+J)$$
$$(r+I) \otimes (s+J) \longmapsto rs + (I+J)$$

3. (a) Let D be a right R–module. Prove that this map is a (well-defined) additive covariant functor:

$$D \otimes_R - : R\text{–}\underline{\text{Mod}} \longrightarrow \underline{\text{Ab}}$$
$$M \longmapsto D \otimes_R M$$
$$[\phi : M \to N] \longmapsto \begin{bmatrix} \phi_* : D \otimes_R M \longrightarrow D \otimes_R N \\ \phi_*(d \otimes m) = d \otimes \phi(m) \end{bmatrix}$$

   By definition, the functor is *additive* if the maps $\text{Hom}_R(M, N) \to \text{Hom}_{\mathbb{Z}}(D \otimes_R M, D \otimes_R N)$ are maps of abelian groups for all objects $M, N \in R$–$\underline{\text{Mod}}$.
   (b) **(The tensor-Hom adjunction.)** Let S, R by rings. Let A be an (S, R)–bimodule, B a left R–module, and C a left S–module. Prove that there is a (well-defined) isomorphism of abelian groups

$$\text{Hom}_S(A \otimes_R B, C) \xrightarrow{\cong} \text{Hom}_R\big(B, \text{Hom}_S(A, C)\big)$$
$$\big[f : a \otimes b \longmapsto f(a \otimes b)\big] \longmapsto \Big[b \longmapsto [a \longmapsto f(a \otimes b)]\Big]$$

   It turns out that this bijection is *natural*, so the functors $A \otimes_R -$ and $\text{Hom}_S(A, -)$ are adjoints.

4. **Bonus (Optional). (The functor $D \otimes_R -$ is right exact.)** Let F be an additive covariant functor from R–$\underline{\text{Mod}}$ to S–$\underline{\text{Mod}}$. Then F is called *exact* if for any short exact sequence of R–modules

$$0 \longrightarrow A \xrightarrow{\psi} B \xrightarrow{\varphi} C \longrightarrow 0,$$

   the image

$$0 \longrightarrow F(A) \xrightarrow{F(\psi)} F(B) \xrightarrow{F(\varphi)} F(C) \longrightarrow 0$$

   is exact. A weaker condition: the functor F is called *right exact* if the resulting sequence is always exact on the right, that is, the sequence

$$F(A) \xrightarrow{F(\psi)} F(B) \xrightarrow{F(\varphi)} F(C) \longrightarrow 0$$

   is exact. Let R be any ring, and D a right R–module.
   (a) Show by example that the functor $D \otimes_R - : R$–$\underline{\text{Mod}} \to \mathbb{Z}$–$\underline{\text{Mod}}$ may *not* be exact.
   (b) Show that $D \otimes_R -$ is right exact. (This result turns out to be a useful computational tool).
   Hint: Dummit-Foote 10.5 Theorem 39. An alternate proof on p402 uses the tensor-Hom adjunction.
   (c) Show that, if k is a field and V a k–vector space, the functor $V \otimes_k -$ is exact.