Reading: Dummit–Foote Ch 10.4, 11.5, 18.1.

## Summary of definitions and main results

**Definitions we've covered:** $R$–algebra, $k^{th}$ tensor power $T^k(M)$, tensor algebra $T^*(M)$, $k^{th}$ symmetric power $\text{Sym}^k(M)$, symmetric algebra $\text{Sym}^*(M)$, $k^{th}$ exterior power $\bigwedge^k M$, exterior algebra $\bigwedge^* M$, group ring, (linear) representation, degree of a representation, faithful representation, trivial representation, permutation representation, regular representation, homomorphism and isomorphism of representations, $G$-equivariant map, intertwiner, minimal polynomial of a linear map.

**Main results:** using right exactness to compute tensor products, construction & universal properties for tensor, symmetric, and exterior powers and algebras, equivalent definitions of a group representation.

## Warm-Up Questions

1. (a) We defined how to form the tensor product $M \otimes_R N$ of a right $R$–module $M$ and a left $R$–module $N$. What would go wrong with this construction if $M$ instead had the structure of a left $R$–module?
   (b) Show that if $M$ is an $(S, R)$–bimodule and $N$ a left $R$–module, the tensor product $M \otimes_R N$ has the structure of an $S$–module. Why must the left action of $S$ and the right action of $R$ on $M$ commute?
2. Verify that the tensor product of maps respects composition:
$$(\tilde{\phi} \otimes \tilde{\psi}) \circ (\phi \otimes \psi) = (\tilde{\phi} \circ \phi) \otimes (\tilde{\psi} \circ \psi).$$
3. Let $\phi : M \to M'$ be a map of right $R$–modules, and $\psi : N \to N'$ be a map of left $R$–modules.
   (a) Show by example that even if $\phi$ and $\psi$ both inject, their tensor product $\phi \otimes \psi$ may not be injective.
   (b) Show that if $\phi$ and $\psi$ are both surjective, then their tensor product $\phi \otimes \psi$ will be surjective.
   (c) Show that if $\phi$ and $\psi$ are both isomorphisms, then their tensor product $\phi \otimes \psi$ will be an isomorphism *Hint:* Isomorphisms have inverses. Use Warm-Up Problem 2.
4. Fill in the details of the proof of that the tensor product associates (Dummit-Foote 10.4 Theorem 14).
5. Let $M$ be a right $R$–module and $N_1, \ldots, N_n$ a set of left $R$–modules. Verify that the tensor product distributes over direct sums (Dummit–Foote 10.4 Theorem 17). There is a unique group isomorphism
$$M \otimes_R (N_1 \oplus \cdots \oplus N_n) \cong (M \otimes_R N_1) \oplus \cdots \oplus (M \otimes_R N_n).$$
   Conclude that if $N$ is a left $R$–module, $R^n \otimes_R N \cong N^n$ .
6. Show that the following alternate definition of an $R$–algebra $A$ is equivalent to the one from class. Given a commutative ring $R$, an *$R$–algebra* $A$ is an $R$–module $A$ with a ring structure such that the multiplication map $A \times A \to A$ is $R$–bilinear.
7. Let $R$ be a commutative ring, and $M$ and $R$–module.
   (a) Verify that, if $R$ does not have characteristic 2, then the submodule
$$\langle m_1 \otimes m_2 \otimes \cdots \otimes m_k \mid m_i = m_j \text{ for some } i \neq j \rangle \subseteq T^k M$$
   defining the exterior power $\bigwedge^k M$ is equal to the submodule
$$\langle m_1 \otimes m_2 \otimes \cdots \otimes m_k - \text{sign}(\sigma) m_{\sigma(1)} \otimes m_{\sigma(2)} \otimes \cdots \otimes m_{\sigma(k)} \mid \sigma \in S_k \rangle.$$
   (b) Are these submodules the same when $R$ has characteristic 2?

8. Let $R$ be a commutative ring and $M$ and $R$–module. Verify the universal properties for the $R$–modules

$$\text{(a) } \mathrm{T}^k(M) \qquad \text{(b) } \mathrm{Sym}^k(M) \qquad \text{(c) } \bigwedge^k(M)$$

and for $R$–algebras

$$\text{(d) } \mathrm{T}^*(M) \qquad \text{(e) } \mathrm{Sym}^*(M)$$

9. Let $G$ be a group and $V$ an $\mathbb{F}$-vector space. Show that the following are all equivalent ways to define a (linear) representation of $G$ on $V$.
   i. A group homomorphism $G \to \mathrm{GL}(V)$.
   ii. A group action (by linear maps) of $G$ on $V$.
   iii. An $\mathbb{F}[G]$–module structure on $V$.

10. Let $R$ be a commutative ring. Show that the group ring $R[\mathbb{Z}] \cong R[t, t^{-1}]$. Show that $R[\mathbb{Z}/n\mathbb{Z}] \cong R[t]/\langle t^n - 1 \rangle$.What is the group ring $R[\mathbb{Z}^n]$? The group ring $R[\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}]$?

11. Let $\phi : G \to GL(V)$ be any group representation. What is the image of the identity element in $GL(V)$?

12. Compute the sum and product of $(1 + 3e_{(1\,2)} + 4e_{(1\,2\,3)})$ and $(4 + 2e_{(1\,2)} + 4e_{(1\,3)})$ in the group ring $\mathbb{Q}[S_3]$.

13. Let $G$ be a group and $R$ a commutative ring. Show that $R[G]$ is commutative if and only if $G$ is abelian.

14. Given any representation $\phi : G \to GL(V)$, prove that $\phi$ defines a faithful representation of $G/\ker(\phi)$.

15. (a) Find an explicit isomorphism $T$ between the following two representations of $S_2$.

$$S_2 \to GL(\mathbb{R}^2) \qquad\qquad S_2 \to GL(\mathbb{R}^2)$$
$$(1\ 2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad (1\ 2) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

   Give a geometric description of the action and the bases for $\mathbb{R}^2$ associated to each matrix group.

   (b) Prove that the following two representations of $S_2$ are not isomorphic.

$$S_2 \to GL(\mathbb{R}^2) \qquad\qquad S_2 \to GL(\mathbb{R}^2)$$
$$(1\ 2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad\qquad (1\ 2) \mapsto \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

16. Fix an integer $n > 0$. Recall the following example from class: The symmetric group $S_n$ acts on $\mathbb{C}^n$ by permuting a basis $e_1, e_2, \ldots, e_n$. We saw that this representation has two subrepresentations,

$$D = \mathrm{span}_{\mathbb{C}}(e_1 + e_2 + \cdots e_n) \qquad \text{and} \qquad U = \{a_1e_1 + a_2e_2 + \cdots + a_ne_n \mid a_1 + a_2 + \cdots a_n = 0\}.$$

   Show that, as a $\mathbb{C}S_n$–module, $\mathbb{C}^n$ is the direct sum $\mathbb{C}^n \cong D \oplus U$.

17. Let $A$, $B$, $C$ be linear maps $V \to V$, with $C$ invertible. Verify the following properties of the trace.
   (a) $\mathrm{Trace}(CAC^{-1}) = \mathrm{Trace}(A)$ (so trace does not depend on choice of basis or matrix representing $A$).
   (b) $\mathrm{Trace}(cA + B) = c\mathrm{Trace}(A) + \mathrm{Trace}(B)$ for any scalar $c$.
   (c) $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$ but $\mathrm{Trace}(AB) \neq \mathrm{Trace}(A)\mathrm{Trace}(B)$ in general.
   (d) $\mathrm{Trace}(A) = \mathrm{Trace}(A^T)$.
   (e) $\mathrm{Trace}(\mathrm{Id}_V) = \dim(V)$.
   (f) $\mathrm{Trace}(A)$ is the sum of the eigenvalues of $A$ (with algebraic multiplicity).
   (g) If $A$ has characteristic polynomial $p_A(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\mathrm{Trace}(A) = a_{n-1}$.
   (h) If $V = U \oplus W$ and $U, W$ are stabilized by $A$, then $\mathrm{Trace}(A) = \mathrm{Trace}(A|_U) + \mathrm{Trace}(A|_W)$.

# Assignment Questions

1. (a) Use the right-exactness of the functor $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} -$ and the short exact sequence of $\mathbb{Z}$–modules

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

to (re)compute the abelian group $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$.

(b) Let $k$ be a field and let $R = k[x, y]$. Using any method you prefer, give simple descriptions of the following $R$–modules, and determine their dimensions over $k$.

$$\frac{R}{(x)} \otimes_R \frac{R}{(x-y)} \qquad \frac{R}{(x)} \otimes_R \frac{R}{(x-1)} \qquad \frac{R}{(y-1)} \otimes_R \frac{R}{(x-y)}$$

2. Let $R$ be a commutative ring and $M$ and $R$–module.

(a) For any commutative ring $R$ and $R$–module $M$, show that the $R$–module $T^*M := \bigoplus_{i=0}^{\infty} M^{\otimes i}$ has the structure of an $R$–algebra. Verify that this algebra may not be commutative.

(b) A similar proof shows that $\mathrm{Sym}^*M := \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(M)$ and $\bigwedge^* M := \bigoplus_{i=0}^{\infty} \bigwedge^i M$ are $R$–algebras. You do not need to give a full proof, but verify that multiplication is well-defined for these spaces (it is independent of representative of an equivalence class of elements in these quotients).

3. Let $\mathbb{F}$ be a field of characteristic zero and $V$ a vector space over $\mathbb{F}$ with basis $\{x_1, \ldots, x_n\}$.

(a) Verify that $\mathrm{Sym}^k(V)$ is a vector space over $\mathbb{F}$ with basis given by the set of monomials in the variables $\{x_1, x_2, \ldots, x_n\}$ of total degree $k$. (*Remark:* There are $\binom{n+k-1}{n-1}$ such monomials).
*Hint*: To show these elements are linearly independent, it is enough to use the universal property to define a symmetric multilinear map $V^k \to \mathbb{F}$ that factors through $\mathrm{Sym}^k V$ which takes value 1 on one basis element and 0 on all others.

(b) Verify that $\bigwedge^k V$ is isomorphic to the $\mathbb{F}$–vector space with a basis given by elements of the form $x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_k}$ with $i_1 < i_2 < \cdots < i_k$. (*Remark:* There are $\binom{n}{k}$ such elements).

(c) Suppose that $A : V \to V$ is a diagonalizable linear map with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (listed with multiplicity). Compute the eigenvalues of the maps induced by $A$ on $T^k V$, $\mathrm{Sym}^k(V)$, and $\wedge^k V$.

(d) Show that you can identify $\mathrm{Sym}^*V$, and $\bigwedge^* V$ as **direct summands** of $T^*V$ via the (split) maps

$$x_1 x_2 \cdots x_k \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \quad \text{and} \quad x_1 \wedge x_2 \wedge \cdots \wedge x_k \longmapsto \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\sigma(x_1 \otimes x_2 \otimes \cdots \otimes x_k)$$

(We are using the assumption that $\mathbb{F}$ has characteristic zero, so the integer $k!$ is invertible in $\mathbb{F}$.)

(e) Show that $V \otimes_{\mathbb{F}} V \cong \mathrm{Sym}^2(V) \oplus \wedge^2 V$.

(f) Show that if $V$ has dimension at least 2, then $V \otimes_{\mathbb{F}} V \otimes_{\mathbb{F}} V \supsetneq \mathrm{Sym}^3(V) \oplus \wedge^3 V$.

4. Let $G$ be a finite group, and $\mathbb{F}$ a field. You may use properties of the trace without proof.

(a) Let $G \to GL(U)$ be any representation of $G$. Citing facts from linear algebra (which you don't need to prove), explain why the trace of the matrix representing a given element $g \in G$ is well-defined in the sense that it will be the same in any isomorphic representation of $G$.

(b) A *permutation representation* of $G$ on a finite-dimensional $\mathbb{F}$-vector space $V$ is a linear representation $\rho : G \to GL(V)$ in which elements act by permuting some basis $B = \{b_1, \ldots b_m\}$ for $V$. Show that, with respect to the basis $\{b_1, \ldots, b_m\}$, for each element $g \in G$, $\rho(g)$ is represented by an $m \times m$ *permutation matrix*, a square matrix that has exactly one entry 1 in each row and each column, and zero elsewhere. Use this description of matrices $\rho(g)$ to show that the trace of $\rho(g)$ is equal to the number of basis elements $b_i$ fixed by $\rho(g)$.

(c) Our first example of a permutation representation was given by the action of $S_n$ on $\mathbb{F}^n$ by permuting the basis $e_1, \ldots, e_n$. Show, in contrast, that the subrepresentation

$$U = \{a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \mid a_1 + a_2 + \cdots + a_n = 0\} \subseteq \mathbb{F}^n$$

is *not* a permutation representation with respect to any basis for $U$.
*Hint:* Warm-up Question 17(h). What is the trace of an $n$–cycle?

(d) The group ring of $\mathbb{F}[G]$ is a left module over itself. This corresponds to permutation representation of the group $G$ on the underlying vector space $\mathbb{F}[G]$, called the *(left) regular representation* of $G$. Find the degree of this representation. In what basis is this a permutation representation, and how many $G$-orbits does this basis have?

(e) For any $g \in G$, compute the trace of the matrix representing $g$ in the regular representation.

5. Let $V$ be a $\mathbb{C}[x]$–module that is finite dimensional over $\mathbb{C}$, where $x$ acts on $V$ by a $\mathbb{C}$–linear map $T$. According to the structure theorem for finitely generated modules over a PID, we can write

$$V \cong \frac{\mathbb{C}[x]}{\big(p_1(x)\big)} \oplus \frac{\mathbb{C}[x]}{\big(p_2(x)\big)} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{\big(p_k(x)\big)}$$

for some monic polynomials $p_i(x) \in \mathbb{C}[x]$ such that $p_1(x)$ divides $p_2(x)$, $p_2(x)$ divides $p_3(x)$, etc.

The monic polynomial $p_k(x)$ is called the *minimal polynomial* of $T$, and the product $p_1(x)p_2(x)\cdots p_k(x)$ is called the *characteristic polynomial* of $T$. By construction the minimal and characteristic polynomials have the same set of roots (possibly with different multiplicities).

(a) Verify that if $\lambda \in \mathbb{C}$ is a root of $p_i(x)$, then $\dfrac{p_i(x)}{(x-\lambda)} \in \dfrac{\mathbb{C}[x]}{\big(p_i(x)\big)}$ is an eigenvector of $T$ with eigenvalue $\lambda$.

(b) Suppose that $\mu \in \mathbb{C}$ is *not* a root of $p_k(x)$ (and therefore not a root of $p_i(x)$ for any $i$). Show that $\mu$ is not an eigenvalue of $T$. Conclude that the eigenvalues of $T$ are precisely the roots of the minimal polynomial $p_k(x)$.
*Hint:* Recall that an eigenvector for $\mu$ is a nonzero element of $\ker(T - \mu I)$, where $I$ is the identity matrix. Consider the projection of a $\mu$–eigenspace onto the summand $\frac{\mathbb{C}[x]}{(p_i(x))}$ for each $i$, and notice that the polynomial $(x - \mu)$ is coprime to $p_i(x)$.

(c) Suppose the roots of $p_k(x)$ are distinct, ie, they each occur with multiplicity one. Show that $T$ is diagonalizable. *Hint:* Chinese Remainder Theorem.

(d) Show that $\mathrm{Ann}(V) = \big(p_k(x)\big)$.

(e) Show that $\mathrm{Ann}(V)$ is equal to the set

$$\{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \in \mathbb{C}[x] \mid a_n T^n + a_{n-1}T^{n-1} + \cdots + a_1 T + a_0 I \text{ is the zero map}\}$$

Conclude that if $p(T) = 0$ for some polynomial $p(x)$, every eigenvalue of $T$ is a root of $p(x)$.

(f) Suppose the linear map $T$ has finite order, that is, $T^n = I$ for some $n \in \mathbb{Z}_{\geq 0}$. Show that $T$ is diagonalizable, and that all its eigenvalues are $n^{th}$ roots of unity.

(g) Let $G$ be a finite group of order $n$, and let $\rho : G \to GL(V)$ be a representations of $G$ on a finite dimensional vector space $V$. Conclude that for every $g \in G$ the linear map $\rho(g)$ is diagonalizable, and its eigenvalues are all $n^{th}$ roots of unity.

6. **Bonus (Optional).** Let $\mathbb{C}^d$ be the canonical permutation representation of the symmetric group $S_d$, and consider the induced action on $\bigwedge^k \mathbb{C}^d$. Prove that

$$\frac{1}{d!} \sum_{\sigma \in S_d} \left(\mathrm{Trace}\left(\sigma \curvearrowright \wedge^k \mathbb{C}^d\right)\right)^2 = 2 \qquad \text{for any } d \geq 1 \text{ and } 0 \leq k \leq d-1.$$

We will see that with *character theory*, this result implies that $\bigwedge^k U$ is an *irreducible* $S_d$–representation for all $0 \leq k \leq d-1$.