Reading: Dummit–Foote Ch 18.1, Fulton–Harris Ch 1.1–1.2.

Summary of definitions and main results

Definitions we've covered: simple (or irreducible) module, decomposable module, completely reducible module.

Main results: Schur's lemma; properties of the averaging map; Maschke's theorem; induced $\mathbb{F}[G]$–modules structures on $V \oplus W$, $\mathrm{Hom}_{\mathbb{F}}(V, W)$, V^*, $V \otimes W$, $\wedge^k V$, $\mathrm{Sym}^k(V)$.

Warm-Up Questions

1. Given a group representation $\phi : G \to \mathrm{GL}(V)$ over a field $\mathbb{F}$, prove that the map

$$\begin{aligned} G &\longrightarrow \mathbb{F}^\times = \mathrm{GL}(\mathbb{F}) \\ g &\longmapsto \det(\phi(g)) \end{aligned}$$

defines a degree–1 representation of G.

2. Given an example of a ring R and an R–module M that is:

   (a) irreducible

   (b) reducible, but not decomposable

   (c) decomposable, but not completely reducible

   (d) completely reducible, but not irreducible

3. Let D_{2n} be the dihedral group, the symmetry group of a regular planar polygon with n edges. Draw the polygon in the plane $\mathbb{R}^2$ centred at the origin and with the y–axis as one of its lines of symmetry. Show that the action of D_{2n} on the polygon extends to a linear action of the plane. Verify that this is an irreducible degree–2 representation of D_{2n}.

4. Let V be a representation of a group G, and recall that V^G denotes the set of vectors in V that are fixed pointwise by the action of every group element $g \in G$. Verify that V^G is a linear subspace of V.

5. Complete our proof of Maschke's Theorem: Suppose $\pi_0 : V \to U$ is a projection map; this means $\pi_0(V) \subseteq U$ and $\pi_0|_U$ is the identity on U. Then the map

$$\pi = \frac{1}{|G|} \sum_{g \in G} g \pi_0 g^{-1} : V \to U$$

is also a projection.

6. (a) Let $\mathbb{C}^n = \langle e_1, \ldots, e_n \rangle$ be the canonical representation of the symmetric group S_n by signed permutation matrices. Explicitly describe the action of the *averaging map* on $\mathbb{C}^n$:

$$\begin{aligned} \psi_{av} : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ v &\longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \cdot v \end{aligned}$$

   (b) Suppose v is an element of the diagonal $D = \mathrm{span}_{\mathbb{C}}(e_1 + e_2 + \cdots + e_n)$. What is $\psi_{av}(v)$?

   (c) Suppose v is an element of the standard subrepresentation $U = \{a_1 e_1 + \cdots + a_n e_n \mid \sum a_i = 0\}$. What is $\psi_{av}(v)$? *Hint:* First check $\psi_{av}(v)$ on the basis vectors $v = (e_1 - e_i)$ for U.

   (d) Interpret your answer to the previous questions, given that we know $\psi_{av} : V \to V$ is a linear projection onto V^G.

7. Let V and W be linear representations of a group G over a field $\mathbb{F}$.
   (a) Show that the tensor product $V \otimes_{\mathbb{F}} W$ has a (well-defined) induced diagonal action of G by
   $$g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w).$$
   (b) Show that the tensor power $T^k(V)$ has an induced structure of a G–representation by the map
   $$g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = (g \cdot v_1) \otimes (g \cdot v_2) \otimes \cdots \otimes (g \cdot v_k)$$
   (c) Show that the above action decends to a well-defined action on $\mathrm{Sym}^k(V)$ and $\bigwedge^k(V)$.

8. **(Linear algebra review.)**
   (a) Define what it means for two matrices to be *conjugate* (or *similar*)
   (b) What is the conjugacy class of the zero matrix? The identity matrix? A scalar matrix?
   (c) Explain why two matrices are conjugate if and only if they represent the same linear map with respect to different bases.
   (d) Show that conjugate matrices have the same determinant.
   (e) Show that $(ABA^{-1})^n = AB^nA^{-1}$.

9. **(Linear algebra review.)** Let $A : V \to V$ be a linear map on a finite dimensional vector space V.
   (a) Suppose A is a *block diagonal matrix*, ie, it has square matrices $\mathbf{A}_i$ (its *blocks*) on the diagonal:
   $$A = \begin{bmatrix} \mathbf{A}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{A}_n \end{bmatrix}$$
   $$\left(\text{eg.} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 6 & 4 \end{bmatrix} \text{ has } \mathbf{A}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 4 & 2 \\ 6 & 4 \end{bmatrix}, \right.$$
   $$\left. \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \text{ has } \mathbf{A}_1 = [1], \mathbf{A}_2 = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}, \mathbf{A}_3 = [4] \right)$$
   Explain how the blocks of A correspond to a decomposition of V into a direct sum of subspaces $V = V_1 \oplus \cdots \oplus V_n$ where each V_i is invariant under the action of A. (The matrix A is sometimes called the *direct sum* of its blocks $A = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_n$.)
   (b) Conversely, explain why, if V decomposes into a direct sum of subspaces that are invariant under A, then the corresponding matrix for A will be block diagonal. (What are the sizes of the blocks?)
   (c) Observe that $\mathrm{Trace}(A) = \mathrm{Trace}(\mathbf{A}_1) + \cdots + \mathrm{Trace}(\mathbf{A}_n)$, and $\mathrm{Det}(A) = \mathrm{Det}(\mathbf{A}_1) \cdots \mathrm{Det}(\mathbf{A}_n)$.
   (d) What is the product of two block diagonal matrices (assuming blocks of the same sizes)?
   (e) Show that for any exponent $p \in \mathbb{Z}_{\geq 0}$, the matrix A^p is block diagonal with blocks $\mathbf{A}_1^p, \ldots, \mathbf{A}_m^p$.

10. **(Linear Algebra Review).** Let V be a finite dimensional vector space over $\mathbb{C}$. For $a \in \mathbb{C}$, write $\overline{a}$ for its complex conjugate. Recall that a *Hermitian inner product* on V is a function
    $$\langle -, - \rangle : V \times V \to \mathbb{C}$$
    satisfying the following properties:
    (1) (Conjugate symmetry)
    $$\langle x, y \rangle = \overline{\langle y, x \rangle} \qquad \forall x, y \in V$$

(2) (Linearity in the first coordinate)

$$\langle ax, y\rangle = a\langle x, y\rangle \qquad \text{and} \qquad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle \qquad \forall x, y, z \in V, a \in \mathbb{C}$$

(3) (Positive definiteness)

$$\langle x, x\rangle \geq 0 \qquad \text{and} \qquad \langle x, x\rangle = 0 \Rightarrow x = 0 \qquad \forall x \in V$$

Observe that (1) and (2) imply that the Hermitian inner product is *antilinear* in the second coordinate:

$$\langle x, ay\rangle = \overline{a}\langle x, y\rangle \qquad \text{and} \qquad \langle x, y+z\rangle = \langle x, y\rangle + \langle x, z\rangle \qquad \forall x, y, z \in V, a \in \mathbb{C}$$

Remark: Compared to the bilinear form $(-,-)$ we studied when we defined dual spaces, the Hermitian inner product $\langle -,-\rangle$ has the advantage that it is positive definite, but the disadvantage that it is not linear in the second argument. These two definitions coincide when we work over $\mathbb{R}$ (instead of $\mathbb{C}$).

(a) Suppose that there is set of vectors $e_1, e_2, \ldots, e_n$ in V that is *orthonormal* with respect to the inner product $\langle -,-\rangle$. This means

$$\langle e_i, e_j\rangle = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Prove these vectors are linearly independent, and therefore form a basis for the space they span. (NB: We can always use the *Gram-Schmidt algorithm* to construct an orthonormal basis for V.)

(b) Let $v = a_1e_1 + \cdots + a_ne_n$ and $w = b_1e_1 + \cdots + b_ne_n$ be elements of V. Compute $\langle v, w\rangle$. Show in particular that

$$\langle v, e_i\rangle = a_i \qquad \text{and} \qquad \langle v, v\rangle = |a_1|^2 + |a_2|^2 + \cdots + |a_n|^2.$$

(c) Show that the function

$$\begin{aligned} ||-|| : V &\longrightarrow \mathbb{R}_{\geq 0} \\ ||v|| &= \sqrt{\langle v, v\rangle} \end{aligned}$$

defines a norm on V, and hence the function

$$\begin{aligned} d : V \times V &\longrightarrow \mathbb{R}_{\geq 0} \\ d(v, w) &= ||v - w|| \end{aligned}$$

defines a metric on V.

(d) Suppose that $v = a_1e_1 + \cdots + a_ne_n$ for **nonnegative integer** coefficients a_i. Show that

$$\langle v, v\rangle = a_1^2 + a_2^2 + \cdots + a_n^2,$$

and conclude that $\langle v, v\rangle = 1$ if and only if $v = e_i$ for some i.

(e) Suppose you have a function $\langle -,-\rangle : V \times V \to \mathbb{C}$ which you know satisfies the conjugate-symmetry and linearity properties of an inner product. Show that, if V has an basis that is orthonormal with respect to the function, then it must be positive definite.

Assignment Questions

For this assignment, you may quote basic results from linear algebra (including facts about matrix inverse, transpose, trace, and determinant) and basic facts about complex conjugation without proof.

Notation. Given a group representation $\varphi : G \to \mathrm{GL}(V)$ on a finite dimensional vector space V, for any $g \in G$, write $\chi_V(g)$ to denote the trace of the linear map $\varphi(g)$.

1. (a) Let G be a finite **abelian** group, and V a finite-dimensional complex representation of G. Show that the image of G in $GL(V)$ is *simultaneously diagonalizable*, that is, there is some basis for V with respect to which every matrix is diagonal. Conclude that V decomposes into a direct sum of 1-dimensional G–representations.

   (b) It follows that all irreducible complex G–representations are 1–dimensional. Let C_n denote the cyclic group of order n. Find n irreducible degree–1 representations of C_n. Show that they are non–isomorphic, and comprise a complete list of its irreducible representations.

2. Let G be a group with a linear actions on finite dimensional $\mathbb{F}$–vector spaces V and W, given by $\rho : G \to GL(V)$ and $\varphi : G \to \mathrm{GL}(W)$.

   (a) Show that the $\mathbb{F}$–vector space of linear maps $\mathrm{Hom}_{\mathbb{F}}(V, W)$ inherits the structure of a G–representation, where an element g in G acts by

   $$f \longmapsto \Big[v \mapsto \varphi(g) \Big(f\Big(\rho(g)^{-1}(v)\Big)\Big) \Big] \qquad \forall\, f \in \mathrm{Hom}_{\mathbb{F}}(V, W),\ g \in G$$

   (b) Consider the special case of this construction when $W = \mathbb{F}$ is the trivial G–representation, so $\mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F}) = V^*$. In this case, the induced representation of G on V^* is called the *dual representation* ρ^* of ρ, and given by

   $$\rho^*(g) : f \longmapsto \Big[v \mapsto f\Big(\rho(g)^{-1}(v)\Big) \Big] \qquad \forall\, f \in V^*,\ g \in G.$$

   If A is the matrix representing the action of a group element $g \in G$ on V with respect to a basis B, show that the matrix for g on V^* with respect to the dual basis B^* is given by $(A^{-1})^T$, the inverse transpose of A.

   (c) Now suppose that $k = \mathbb{C}$ and G is finite, and let $g \in G$. Prove that $\chi_{V^*}(g)$ is the complex conjugate of $\chi_V(g)$. *Hint:* What are its eigenvalues?

   (d) Let $V \cong \mathbb{C}^d$ be the canonical permutation representation of the symmetric group S_d. Show V is *self-dual*, that is, $V \cong V^*$ as S_d–representations.

   (e) Consider the 3 non-isomorphic degree–1 complex C_3–representations from Problem 1(b). Show that one of these is self-dual, and the other two are not.

3. Let G be a finite group and $\mathbb{F}$ a field.

   (a) Suppose that A and B are finite order (therefore diagonalizable) endomorphisms of finite dimensional vector spaces V and W over an algebraically closed field $\mathbb{F}$. Show that the trace of $A \otimes B$ on $V \otimes_{\mathbb{F}} W$ is the product $\mathrm{Trace}(A)\mathrm{Trace}(B)$.
   Remark: This result also holds when A and B are not diagonalizable, and can be proven (with a little more effort) by considering the bases for V and W putting A and B into Jordan canonical form. It can also be proven for arbitrary fields, using extension of scalars to the algebraic closure.

   (b) Let V and W be finite-dimensional representations of G over $\mathbb{C}$. Conclude that

   $$\chi_{V \otimes_{\mathbb{C}} W}(g) = \chi_V(g)\chi_W(g) \qquad \text{for all } g \in G.$$

   (c) Let $\mathbb{F}$ be any field, and again let V and W be finite-dimensional representations of G over a field $\mathbb{F}$. Construct an isomorphism of G–representations $\mathrm{Hom}_{\mathbb{F}}(V, W) \cong V^* \otimes_{\mathbb{F}} W$. This isomorphism should be *natural*, that is, its definition should not require a choice of basis for V or W.
   (It's okay if you choose bases in the proof that it is an isomoprhism).

(d) Suppose $\mathbb{F} = \mathbb{C}$. Show that

$$\chi_{\mathrm{Hom}_{\mathbb{C}}(V,W)}(g) = \overline{\chi_V(g)}\chi_W(g) \qquad \text{for all } g \in G.$$

Remark: This will be a key result in our development of character theory!

4. (a) Let R be a commutative ring, and let S be an R–algebra. Given S–modules U, V, W, show that there are isomorphisms of R–modules

$$\mathrm{Hom}_S(V \oplus U, W) \cong \mathrm{Hom}_S(V, W) \oplus \mathrm{Hom}_S(U, W)$$

$$\mathrm{Hom}_S(V, U \oplus W) \cong \mathrm{Hom}_S(V, U) \oplus \mathrm{Hom}_S(V, W)$$

(b) Let $\{V_i\}$ be a finite set of irreducible G–representations over $\mathbb{C}$. Let

$$U = \bigoplus V_i^{\oplus a_i} \qquad \text{and} \qquad W = \bigoplus V_j^{\oplus b_j} \qquad \text{for } a_i, b_j \in \mathbb{Z}_{\geq 0}.$$

Compute $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(U, W)$.

5. Let V be a $\mathbb{C}[x]$–module that is finite dimensional over $\mathbb{C}$, where x acts on V by a $\mathbb{C}$–linear map T. Recall from the last homework that we can write

$$V \cong \frac{\mathbb{C}[x]}{\big(p_1(x)\big)} \oplus \frac{\mathbb{C}[x]}{\big(p_2(x)\big)} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{\big(p_k(x)\big)}$$

for some monic polynomials $p_i(x) \in \mathbb{C}[x]$ such that $p_1(x)$ divides $p_2(x)$, $p_2(x)$ divides $p_3(x)$, etc.

(a) Explain why V can also be further decomposed as a direct sum

$$V \cong \frac{\mathbb{C}[x]}{(x-\lambda_1)^{k_1}} \oplus \frac{\mathbb{C}[x]}{(x-\lambda_2)^{k_2}} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{(x-\lambda_d)^{k_d}}$$

for (not necessarily distinct) scalars $\lambda_i \in \mathbb{C}$ and positive powers k_i.

(b) Consider a T–invariant subspace $\dfrac{\mathbb{C}[x]}{(x-\lambda)^k}$ of V. Write down the matrix for the action of T on this subspace with respect to the ordered basis

$$(x-\lambda)^{k-1}, \quad (x-\lambda)^{k-2}, \quad \cdots \quad (x-\lambda), \quad 1.$$

This matrix is called a *Jordan block* of T with eigenvalue λ.

(c) Verify that λ is the only eigenvalue of this Jordan block, and that the associated eigenspace is 1–dimensional.

(d) Conclude that T is not diagonalizable unless the roots of its minimal polynomial are distinct.

6. **Bonus (Optional).** Let V be a finite-dimensional irreducible representation of a finite group G. Prove that (up to scalars) there is a unique Hermitian inner product on V preserved by G.