Reading: Fulton–Harris Chapter 2.1–2.4.

Summary of definitions and main results

Definitions we've covered: V^G, isotypic component, class function, character, character table, the inner product $\langle -, - \rangle_G$

Main results: decomposition into irredcible representations is unique, irreducible characters form a basis for the space of class functions, orthogonality relations,

Warm-Up Questions

1. Suppose that V is a finite dimensional vector space over $\mathbb{F}$, and $T : V \to V$ is a diagonalizable linear map. Show that the restriction of T to any T–invariant subspace $W \subseteq V$ will also be diagonalizable, and therefore W must be a direct sum of eigenspaces of T.

2. Let G be a finite group. Show that an $\mathbb{F}[G]$–module V is finitely generated if and only if it is finite dimensional. What if G is infinite?

3. Let G be a finite group and $\phi : G \to GL(V)$ a G–representation over a field $\mathbb{F}$ with character $\chi_V : G \to \mathbb{F}$. Prove that if V is 1-dimensional, then $\chi_V = \phi$. Show by example that if V is at least 2 dimensional, χ_V may not be a group homomorphism.

4. Let V, W be two representations of a group G, and let U_i be an irreducible G–representation. Let $T : V \to W$ be a G–equivariant map. Explain and prove the sense in which T must respect the isotypic component of U_i in V and W.

5. Let G be a finite group and U an irreducible G–representation over $\mathbb{C}$.
   (a) Show that the G–representation $V \cong U \oplus U$ has infinitely many distinct direct sum decompositions into two copies of U.
   (b) Describe the $\mathbb{C}$–vector space of G–equivariant maps $\mathrm{Hom}_{\mathbb{C}[G]}(U^{\oplus a}, U^{\oplus b})$.
   (c) Which of the maps $\mathrm{Hom}_{\mathbb{C}[G]}(U^{\oplus a}, U^{\oplus b})$ are isomorphisms?

6. Let G be a finite group. Verify that $\langle -, - \rangle_G$ satisfies that conjugate symmetry, linearity, and positive definiteness properties that define an inner product.

7. Let G be a finite group.
   (a) State the formula for the inner product on complex-valued class functions of G.
   (b) Let $U = \sum_i V_i^{\oplus a_i}$ and $W = \sum_j V_j^{\oplus b_j}$ for distinct irreducible representations V_i. Compute $\langle \chi_W, \chi_U \rangle_G$.
   (c) Explain why the following results about character theory hold.
      (i) Characters of irreducible representations are orthonormal.
      (ii) Characters of irreducible representations are linearly independent.
      (iii) The number of irreducible representations is at most the number of conjugacy classes of G.
      (iv) A G–representation V is irreducible if and only if $\langle \chi_V, \chi_V \rangle_G = 1$.
      (v) A representation V is determined up to isomorphism by its character.

8. Let G be a finite group. Prove that a complex-valued class function on G is a character if and only if it is a nonnegative integer linear combination of irreducible characters.

9. Let G be a finite group. Prove that the dimension of the space of class functions $G \to \mathbb{F}$ over $\mathbb{F}$ is equal to the number of conjugacy classes of G.

10. Let G be a finite group. We saw in class that, as a module over itself, $\mathbb{C}[G] \cong \bigoplus_i V_i^{\oplus \dim_{\mathbb{C}}(V_i)}$, where $\{V_i\}$ is a complete set of non-isomorphic irreducible representations of G. What is the multiplicity of the trivial representation in $\mathbb{C}[G]$? Find a basis for this subrepresentation.

11. Let G be a group, and V and U be irreducible complex representations of G.
    (a) Show by example that $U \otimes_{\mathbb{C}} V$ may or may not be an irreducible G–representation.
    (b) Prove that if U is 1-dimensional, then $U \otimes_{\mathbb{C}} V$ is an irreducible G–representation.

12. Let G be a finite group.
    (a) If $\phi : G \to \mathrm{GL}(V)$ is a G–representation, prove that $\phi(g) : V \to V$ is G–equivariant if and only if $\phi(g)$ is central in $\phi(G)$.
    (b) Let χ be an irreducible character of G. Prove that for every element g in the center of G, $\chi(g) = \xi\chi(1)$, where ξ is a root of unity in $\mathbb{C}$.

13. Recall the character table for the complex representations of the symmetric group S_3.

	$(\bullet)(\bullet)(\bullet)$	$(\bullet\bullet)(\bullet)$	$(\bullet\bullet\bullet)$
$\underline{\text{Trv}}$	1	1	1
$\underline{\text{Alt}}$	1	-1	1
$\underline{\text{Std}}$	2	0	-1

    (a) Let $\mathbb{C}^3$ denote the canonical permutation representation of S_3. Compute the character of $\underline{\text{Alt}} \otimes_{\mathbb{C}} \mathrm{Sym}^2\mathbb{C}^3$.
    (b) Use the character table to decompose $\underline{\text{Alt}} \otimes_{\mathbb{C}} \mathrm{Sym}^2\mathbb{C}^3$ as a sum of irreducible representations (in the sense of finding the multiplicity of each irreducible representation in the decomposition).
    (c) Verify that the orthogonality relations hold for this character table.

14. Find two non-isomorphic S_3–representations that are the same dimension. Explain why dimension is an isomorphism invariant of G–representations, but is not sufficient to distinguish non-isomorphic representations.

15. For $n \geq 2$, let $\mathbb{C}^n$ be the canonical permutation representation of S_n.
    (a) Prove that $\langle \mathbb{C}^n, \mathbb{C}^n \rangle_{S_n} = 2$.
    (b) Use this result to conclude that the standard representation is irreducible for every $n \geq 2$.

16. (a) Compute the character table of the cyclic group $G = \mathbb{Z}/4\mathbb{Z}$,
    (b) Verify the orthogonality relations on the row and columns of the character table.
    (c) Compute the character of $\bigwedge^3 \mathbb{C}[G]$, and determine its decomposition into irreducible characters.

17. Let G be a finite group and C be its character table (of all irreducible characters).
    (a) Show that the "orthogonality of characters" result is equivalent to the statement that the matrix C satisfies the relation $\overline{C}DC^T = I$ for a certain diagonal matrix D. What is D?
    (b) Conclude from this equation that $C^T\overline{C} = D^{-1}$. Use this equation to derive the second orthogonality result for characters.
    (c) Explicitly verify the relations $\overline{C}DC^T = I$ and $C^T\overline{C} = D^{-1}$ for the character table for S_3.

18. Prove that the character table is an invertible matrix.

Assignment Questions

1. Let G be a finite group. In this question we will describe the ring structure on the group ring $\mathbb{C}[G]$. Let $V_1, \dots, V_k$ denote a complete list of non-isomorphic irreducible complex G–representations.
   (a) The action of G on a representation V is equivalent to the data of a map of rings $\mathbb{C}[G] \to \mathrm{End}_{\mathbb{C}}(V)$, so we obtain a map of rings $\mathbb{C}[G] \to \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{C}}(V_i)$. Show that this map is injective.
   Hint: First show that the regular representation is faithful.
   (b) Conclude (by a dimension count) that there is an isomorphism of rings $\mathbb{C}[G] \cong \bigoplus_{i=1}^k \mathrm{End}_{\mathbb{C}}(V_i)$

2. (a) Compute the character table for the symmetric group S_5 over $\mathbb{C}$.
   (b) Let $\mathbb{C}^5$ denote the canonical permutation representation of S_5. Use the character table to find the decomposition of $\mathrm{Sym}^2\mathbb{C}^5$ into irreducible S_5–representations.

3. (**Induced representations**) Suppose $H \subseteq G$ are finite groups, and k is a field. Given a finite dimensional G–representation W, we can restrict the action of G to the action of $H \subset G$. The resulting H–representation is denoted $\mathrm{Res}_H^G W$.

   Conversely, given a finite dimensional group representation V of H over k (viewed as a $k[H]$–module), we can construct a representation of G by extension of scalars. Since $k[H]$ is a subring of $k[G]$, we may view $k[G]$ as a right $k[H]$–module. Define a $k[G]$–module, called the *induced representation* $\mathrm{Ind}_H^G V$, by

$$\mathrm{Ind}_H^G V := k[G] \otimes_{k[H]} V.$$

   (a) Cite properties of the tensor product to show that

$$\mathrm{Ind}_H^G(U \oplus U') \cong \mathrm{Ind}_H^G U \oplus \mathrm{Ind}_H^G U' \qquad \text{and} \qquad \mathrm{Ind}_K^G(\mathrm{Ind}_H^K V) \cong \mathrm{Ind}_H^G V$$

   for any representations U, U' of H or subgroups $H \subseteq K \subseteq G$.
   (b) Let G/H be the set of left cosets of G in H, and let $\{\sigma_i\}$ be a set of representatives of each coset. This means for each $g \in G$ and $\sigma_i \in G/H$, there is some $h \in H$ and $\sigma_j \in G/H$ such that $g\sigma_i = \sigma_j h$. Show that $\mathrm{Ind}_H^G V = k[G] \otimes_{k[H]} V$ is isomorphic to the G–representation

$$\bigoplus_{\sigma_i \in G/H} \sigma_i V$$

   where $\sigma_i V := \{\sigma_i v \mid v \in V\}$ has an action of G by $g(\sigma_i v) = \sigma_j h(v)$.
   (c) Given an G–representation W and H–representation V, find the degrees of $\mathrm{Res}_H^G W$ and $\mathrm{Ind}_H^G V$.
   (d) What representation is $\mathrm{Ind}_H^G V$ when H is the trivial group and $V \cong k$ the trivial representation?
   (e) Let $G = S_n$, $H = S_{n-1}$, and V be the degree 1 trivial S_{n-1}–representation. What is $\mathrm{Ind}_{S_{n-1}}^{S_n} V$?
   (f) **(Ind-Res adjunction)** Prove that induction satisfies the following universal property: If U is any representation of G, then any map of $k[H]$–modules $\phi : V \to \mathrm{Res}_H^G U$ can be promoted uniquely to a map of $k[G]$–modules $\Phi : \mathrm{Ind}_H^G V \to U$, such that Φ restricts to the map ϕ on pure tensors of the form $1 \otimes v \in k[G] \otimes_{k[H]} V$. Moreover, every $k[G]$–module map $\mathrm{Ind}_H^G V \to U$ arises in this way. In other words, there is a natural identification of k–modules

$$\mathrm{Hom}_{k[H]}(V, \mathrm{Res}_H^G U) \cong \mathrm{Hom}_{k[G]}(\mathrm{Ind}_H^G V, U).$$

   Hint: It suffices to show this is a special case of the tensor–Hom adjunction from Homework #5. (It is a fact that you do not need to prove that this isomorphism of abelian groups is k–linear).
   (g) **(Frobenius Reciprocity)** Conclude that for finite dimensional representations over $\mathbb{C}$,

$$\langle \chi_{\mathrm{Res}_H^G U}, \chi_V \rangle_H = \langle \chi_U, \chi_{\mathrm{Ind}_H^G V} \rangle_G.$$

   Show in particular that if V and U are irreducible representations of H and G, respectively, then the multiplicity of the $k[H]$–representation V in $\mathrm{Res}_H^G U$ is equal to the multiplicity of the $k[G]$–representation U in $\mathrm{Ind}_H^G V$.

4. Let V be an irreducible complex representation of a finite group G, and let H be an index-2 subgroup of G.
   (a) Prove that $\mathrm{Res}_H^G V$ consists of either one or two irreducible H–representations. Prove moreover that the second case occurs if and only if $V \cong V \otimes_{\mathbb{C}} U$, where U is the 1-dimensional nontrivial representation $G \to G/H \cong \{\pm 1\} \subseteq GL(\mathbb{C})$.
   (b) Suppose a group G has an abelian subgroup of index 2. Show that any irreducible representation of G has degree at most 2.
   (c) Conclude that each irreducible complex representation of a dihedral group must have degree 1 or 2.

5. Let $T : V \to V$ be a linear map on a n-dimensional $\mathbb{F}$–vector space V. Recall from the last assignment that you found a basis $e_1, \ldots, e_n$ for V so that T is a sum of Jordan blocks. This is called the *Jordan canonical form* of T. Let I denote the identity matrix.

   Recall that an *eigenvector* v of T with *eigenvalue* λ is defined to be a nonzero element of $\ker(\lambda I - T)$, and that the *eigenspace* E_λ is defined to be the subspace of V

$$E_\lambda = \ker(\lambda I - T) = \{\text{eigenvectors of } T \text{ with eigenvalue } \lambda\} \cup \{0\}$$

   For an eigenvalue λ of T, define the *algebraic multiplicity* of λ to be the multiplicity of the root $(x - \lambda)$ in the characteristic polynomial of T, and the *geometric multiplicity* to be the $\dim_{\mathbb{F}}(E_\lambda)$.
   (a) Let $J_{\lambda,k}$ denote the $k \times k$ Jordan block with diagonal entry λ. Prove that the characteristic polynomial and minimal polynomial of $J_{\lambda,k}$ are both equal to $(x - \lambda)^k$.
   (b) For any linear map T with eigenvalue λ, show that the geometric multiplicity of λ – the dimension of the eigenspace E_λ – is equal to the number of Jordan blocks with diagonal entry λ in the Jordan canonical form of T.

6. **Bonus (optional).** Compute the character tables for the dihedral groups D_5 (the symmetries of a pentagon) and D_6 (the symmetries of a hexagon).