Reading: Dummit–Foote 18.3, 19.1, 19.3. Fulton–Harris Ch 3.1, 3.3–3.5.

Summary of definitions and main results

Definitions we've covered: induced representations, real and quaternionic structures, generalized eigenspaces.

Main results: character tables for S_4 and A_5, Frobenius reciprocity, Mackey's criterion.

Warm-Up Questions

1. Let U and W be complex representations of a finite group G. Show that $(U \oplus W)^G \cong U^G \oplus W^G$.
2. Suppose that G is a group with N_G conjugacy classes, and H a group with N_H conjugacy classes. Verify that $G \times H$ has $N_G N_H$ conjugacy classes.
3. Verify that a conjugacy class in S_n will break up into two conjugacy classes in A_n if and only if it corresponds to a cycle type where all cycle lengths are odd and distinct.
4. Let G be a finite group and H a subgroup. Let e be the identity element of G.
   (a) Show that $\mathrm{Ind}_H^G \mathbb{C}[H] \cong \mathbb{C}[G]$. Note the special case $\mathrm{Ind}_{\{e\}}^G \mathbb{C} \cong \mathbb{C}[G]$.
   (b) Consider the trivial action of H on $\mathbb{C}$. Show that $\mathrm{Ind}_H^G \mathbb{C}$ is the permutation representation of G on the set of cosets G/H.
5. Use Frobenius reciprocity to perform the following computations.
   (a) Let $C_3 = \{1, (123), (321)\} \subseteq S_3$, and let V be the irreducible trivial C_3–representation. Find the decomposition of the induced S_3–representation $\mathrm{Ind}_{C_3}^{S_3} V$ into irreducible representations.
   (b) Do the same for the irreducible C_3–representation where (123) acts by mulitplication by $e^{\frac{2\pi i}{3}}$.
   (c) Let $C_2 = \{1, (12)\} \subseteq S_3$. Decompose the S_3–representations induced from the trivial and the nontrivial irreducible representations of C_2.
6. Show that the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$ satisfies the polynomial $x^2 - x - 2$. What is its minimal polynomial?
7. Find the characteristic polynomial and the minimal polynomials of the following matrices.
   $$\begin{pmatrix} 3&0&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3 \end{pmatrix} \quad \begin{pmatrix} 3&1&0&0\\0&3&0&0\\0&0&3&0\\0&0&0&3 \end{pmatrix} \quad \begin{pmatrix} 3&1&0&0\\0&3&0&0\\0&0&3&1\\0&0&0&3 \end{pmatrix} \quad \begin{pmatrix} 3&1&0&0\\0&3&1&0\\0&0&3&0\\0&0&0&3 \end{pmatrix} \quad \begin{pmatrix} 3&1&0&0\\0&3&1&0\\0&0&3&1\\0&0&0&3 \end{pmatrix}$$
8. For each of the following $\mathbb{C}[x]$–modules, write the Jordan form of the linear map "multiplication by x". State the minimal and characteristic polynomials.
   $$\frac{\mathbb{C}[x]}{(x-1)^2} \oplus \frac{\mathbb{C}[x]}{(x-1)(x-2)} \qquad \frac{\mathbb{C}[x]}{(x-1)(x-2)(x-3)} \qquad \frac{\mathbb{C}[x]}{(x-1)} \oplus \frac{\mathbb{C}[x]}{(x-1)^2} \oplus \frac{\mathbb{C}[x]}{(x-1)^2}$$
9. Determine all possible Jordan forms for linear maps with characteristic polynomial $(x-1)^3(x-2)^2$.
10. (a) Suppose a complex matrix A satisfies the equation $A^2 = -2A - 1$. What are the possibilities for its Jordan form?
    (b) Suppose a complex matrix A satisfies $A^3 = A$. Show that A is diagonalizable. Would this result hold if A had entries in a field of characteristic 2?
11. Prove that an $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

Assignment Questions

1. **(Real and quaternionic structures.)** Let G be a finite group. All representations are assumed finite dimensional.
   Hint: You are welcome to consult Fulton–Harris Chapter 3.5. Be sure to rephrase and fill in the details of any proof from this section that you wish to use.
   (a) Show that every complex G–representation V has a Hermitian inner product $\langle -, - \rangle$ that is is G-invariant, that is,
   $$\langle gv, gw \rangle = \langle v, w \rangle \qquad \text{for all } v, w \in V \text{ and } g \in G.$$
   Hint: Use the averaging map.
   (b) Let V be a complex G–representation. Prove the isomorphisms of G–representations
   $$(V \otimes_{\mathbb{C}} V)^* \cong V^* \otimes_{\mathbb{C}} V^*.$$
   Conclude that $(V^* \otimes_{\mathbb{C}} V^*)$ is the $\mathbb{C}$–vector space of bilinear forms on V.
   (c) Interpret the decomposition
   $$(V^* \otimes_{\mathbb{C}} V^*) \cong \mathrm{Sym}^2(V^*) \oplus \bigwedge^2 V^*$$
   as a decomposition of the space of bilinear forms on V.
   (d) A representation V of G over $\mathbb{C}$ is called *real* if $V \cong V_0 \otimes_{\mathbb{R}} \mathbb{C}$ for some representation V_0 over $\mathbb{R}$. Show that V is real if and only if V admits a G–equivariant *real structure*, that is, a conjugate-linear map $R : V \to V$ such that $R^2(v) = v$ for all $v \in V$.
   (e) Show that an irreducible complex representation V of G is real if and only if there is a G–invariant nondegenerate symmetric bilinear form $B(-, -)$ on V.
   (f) A representation V of G over $\mathbb{C}$ is called *quaternionic* if V has a G–equivariant conjugate-linear map $J : V \to V$ such that $J^2(v) = -v$ for all $v \in V$. Prove that if V is irreducible then this is equivalent to the existence of a G–invariant nondegenerate bilinear form $H(-, -)$ on V that is *skew-symmetric*, that is,
   $$H(v, w) = -H(w, v) \qquad \text{for all } v, w \in V.$$
   (g) Assume V is irreducible. Interpret the condition that V is real and the condition that V is quaternionic as conditions on the invariants
   $$(V^* \otimes_{\mathbb{C}} V^*)^G \cong \left(\mathrm{Sym}^2(V^*)\right)^G \oplus \left(\bigwedge^2 V^*\right)^G.$$
   (h) **(The Frobenius–Schur indicator.)** Assume V is irreducible. Prove that
   $$\frac{1}{|G|} \sum_{g \in G} \chi_V(g^2) = \begin{cases} 1 & V \text{ is real} \\ -1 & V \text{ is quaternionic} \\ 0 & \text{otherwise.} \end{cases}$$
   Hint: $(V^* \otimes_{\mathbb{C}} V^*)^G \cong \mathrm{Hom}_{\mathbb{C}}(V^*, V)^G \cong \mathrm{Hom}_{\mathbb{C}[G]}(V^*, V)$. Schur's Lemma.
   (i) Prove that the character of a representation V is real if and only if V is either real or quaternionic.

2. Let $T : V \to V$ be a linear map on a n-dimensional $\mathbb{C}$–vector space V. Recall the decomposition of V
   $$V \cong \frac{\mathbb{C}[x]}{(x-\lambda_1)^{k_1}} \oplus \frac{\mathbb{C}[x]}{(x-\lambda_2)^{k_2}} \oplus \cdots \oplus \frac{\mathbb{C}[x]}{(x-\lambda_d)^{k_d}}.$$
   The structure theorem implies that this decomposition is unique up to the order of the factors.
   For an eigenvalue λ of T, let E_λ denote the corresponding eigenspace, and define the *generalized eigenspace of* λ to be the subspace
   $$G_\lambda = \{v \mid (\lambda I - T)^k v = 0 \text{ for some integer } k > 0\} \subseteq V$$

(a) Show (in a sentence) that $E_\lambda \subseteq G_\lambda$.

(b) Show that the generalized eigenspace G_λ of V is precisely the direct sum of submodules of the form $\mathbb{C}[x]/(x-\lambda)^k$ in the decomposition of V.

(c) Conclude that V decomposes into a direct sum of generalized eigenspaces for T, and that the algebraic multiplicity of an eigenvalue λ is equal to sum of the sizes of the corresponding Jordan blocks, which is equal to the dimension of G_λ.

(d) Note as a corollary that dimension of the eigenspace E_λ is no greater than the algebraic multiplicity of λ. Under what conditions are they equal?

(e) Briefly explain how you can compute the Jordan canonical form of a linear map T acting on V (which is uniquely defined up to order of the blocks) by computing its eigenvalues λ, and computing the dimensions of the $\ker(T-\lambda I)^m$ for each eigenvalue λ and $m \leq \dim_{\mathbb{C}}(V)$. **No justification needed.**

(f) State instructions for how to read off the following data from the Jordan canonical form of a linear map T, and state each for the specific map T_0 given below.
You do not need justify instructions or show your computations.

$$T_0 = \begin{bmatrix} 2 & 1 & & & & & & & & \\ & 2 & & & & & & & & \\ & & 2 & 1 & & & & & & \\ & & & 2 & & & & & & \\ & & & & 2 & 1 & & & & \\ & & & & & 2 & & & & \\ & & & & & & 2 & & & \\ & & & & & & & 2 & & \\ & & & & & & & & 3 & 1 \\ & & & & & & & & & 3 \end{bmatrix}$$

(i) The eigenvalues of T (with algebraic multiplicity).

(ii) The determinant of T.

(iii) The characteristic polynomial of T.

(iv) The minimal polynomial of T.

(v) The eigenvalues of T (with geometric multiplicity).

3. **Bonus (Optional).** Let G be a finite group. Show that the dimension of any complex irreducible representation V of G must divide the order of G.
Hint: Dummit–Foote Ch 19.2 Corollary 5. You are welcome to read all relevant portions of Dummit–Foote, but write the solution in your own words and include any missing details.