Midterm Exam II

Math 122
30 May 2017
Jenny Wilson

Name: ______________________________

Instructions: This exam has 5 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers unless directed otherwise. You may cite any results from class or the homeworks without proof, but do give a complete statement of the result you are using.

You have 50 minutes to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	3	
2	5	
3	4	
4	2	
5	6	
Total:	20	

1. (3 points) Recall that the *(left) regular S_3–representation* $\mathbb{C}[S_3]$ is the group ring $\mathbb{C}[S_3]$ viewed as a left module over itself. Let $V \cong \mathbb{C}[S_3]$ be the same $\mathbb{C}$–vector space, but this time with the structure of an S_3–representation induced by the action of S_3 on itself by conjugation (instead of left multiplication).

   i. Compute the character of V, and fill in the corresponding row of the table below.

   First, we recall that the conjugation action of a permutation $\sigma \in S_n$ is by relabelling an element of S_n (as written in cycle notation):

$$\sigma(a_1\, a_2\, \cdots\, a_k)\sigma^{-1} = (\sigma(a_1)\, \sigma(a_2)\, \cdots\, \sigma(a_k)), \qquad a_i \in \{1, 2, \ldots, n\}.$$

   We note moreover that since the action on $\mathbb{C}[S_3]$ is by permuting the basis S_3, the values of the character are given by the number of basis elements fixed by each element. We compute this for an arbitrary representative of each conjugacy class.

   The identity element fixes all 6 permutations in S_3.
   The 2–cycle $(1\,2)$ fixes the 2 basis elements $e, (1\,2)$
   The 3–cycle $(1\,2\,3)$ fixes the 3 basis elements $e, (1\,2\,3), (3\,2\,1)$.

	$(\bullet)(\bullet)(\bullet)$	$(\bullet\,\bullet)(\bullet)$	$(\bullet\,\bullet\,\bullet)$
Trv	1	1	1
Alt	1	-1	1
Std	2	0	-1
V	6	2	3

   ii. Determine the multiplicities of the irreducible constituents of V.

   For each irreducible representation V_i,

$$\text{Multiplicity of } V_i \text{ in } V = \langle \chi_{V_i}, \chi_V \rangle_{S_3} = \frac{1}{|S_3|} \sum_{\text{Cong classes } C} |C| \chi_{V_i}(C)\chi_V(C).$$

$$\text{Multiplicity of } \underline{\text{Trv}} = \tfrac{1}{6}\Big(1(1)(6) + 3(1)(2) + 2(1)(3)\Big) = \tfrac{18}{6} = 3$$

$$\text{Multiplicity of } \underline{\text{Alt}} = \tfrac{1}{6}\Big(1(1)(6) + 3(-1)(2) + 2(1)(3)\Big) = \tfrac{6}{6} = 1$$

$$\text{Multiplicity of } \underline{\text{Std}} = \tfrac{1}{6}\Big(1(2)(6) + 3(0)(2) + 2(-1)(3)\Big) = \tfrac{6}{6} = 1$$

   So $V \cong \underline{\text{Trv}}^{\oplus 3} \oplus \underline{\text{Alt}} \oplus \underline{\text{Std}}$.

2. Let G be a finite group.

   (a) (4 points) Prove that every character of a complex G–representation is real-valued if and only if every element of G is conjugate to its inverse.

   Let $\rho : G \to \mathrm{GL}(V)$ be a complex G–representation of degree n.

   Lemma: $\chi_V(g^{-1}) = \overline{\chi_V(g)}$ for all $g \in G$.

   Proof of Lemma: We proved on homework that the linear map $\rho(g)$ is diagonalizable, and that its eigenvalues $\lambda_1, \dots, \lambda_n$ are roots of unity. By definition

$$\chi_V(g) = \mathrm{Trace}(\rho(g)) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

   But then the eigenvalues of $\rho(g^{-1}) = \rho(g)^{-1}$ are $\lambda_1^{-1}, \dots \lambda_n^{-1}$, and

$$\begin{aligned}\chi_V(g^{-1}) &= \lambda_1^{-1} + \lambda_2^{-1} + \cdots + \lambda_n^{-1}\\ &= \overline{\lambda_1} + \overline{\lambda_2} + \cdots + \overline{\lambda_n}\\ &= \overline{\lambda_1 + \lambda_2 + \cdots + \lambda_n}\\ &= \overline{\chi_V(g)}. \qquad \square\end{aligned}$$

   So suppose any $g \in G$ is conjugate to its inverse. Since χ_V is a class function,

$$\chi_V(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)} \qquad \text{for all } g \in G \text{, so } \chi_V(g) \text{ is real as claimed.}$$

   Suppose conversely that every character χ_V of G is real-valued, so $\chi_V = \overline{\chi_V}$. Let $g \in G$, and V_i an irredcible G–representation. By the Lemma,

$$\chi_{V_i}(g) = \overline{\chi_{V_i}(g^{-1})} = \chi_{V_i}(g^{-1}),$$

   and we see that every irreducible character agrees on g and g^{-1}. But the columns of the character table for G are linearly independent (in fact, orthogonal), so the columns for the conjugacy class of g and the conjugacy class of g^{-1} cannot coincide unless they are the same column. We conclude that g and g^{-1} are conjugate.

   (b) (1 point) Conclude that every finite-dimensional complex G–representation is self-dual if and only if every element of G is conjugate to its inverse.

   We proved that a finite-dimensional complex G–representation V is completely determined by its character, and moreover that $\chi_{V^*} = \overline{\chi_V}$. So V is self-dual if and only if its character is real-valued, and this is true of every character if and only if all elements of G are conjugate to their inverses.

3. (4 points) Let p be a prime and $\mathbb{F}_p$ the field of order p. Let G be a finite group and H a subgroup such that p does not divide the index $m = |G/H|$.

   Suppose V is a G–representation of finite dimension over $\mathbb{F}_p$, with the property that every H–subrepresentation of $\mathrm{Res}^G_H(V)$ has an (H–invariant) direct complement. Show that any G–subrepresentation of V has a (G–invariant) direct complement.

   We will mimic our proof of Maschke's theorem. Let U be a G–subrepresentation of V. By assumption, the short exact sequence of vector spaces

$$0 \longrightarrow U \xrightarrow{\iota} V \longrightarrow V/U \longrightarrow 0$$

   is split when viewed as a sequence of $\mathbb{F}_p[H]$–modules. The Splitting Lemma states that this is equivalent to the existence of an H–equivariant splitting map $\pi_0 : V \to U$ so that $\pi_0 \circ \iota = \mathrm{id}_U$. We will use a variation of our averaging operation to modify π_0 to obtain a splitting map π that is G–equivariant, which (by the Splitting Lemma) proves that the sequence is also split as a sequence of $\mathbb{F}_p[G]$–modules.

   Let $g_1, \ldots, g_m$ be a choice of representatives of the cosets G/H. Since m is invertible in $\mathbb{F}_p$ by assumption, we can define

$$\begin{aligned} \pi : V &\longrightarrow U, \\ v &\longmapsto \frac{1}{m}\sum_{i=1}^{m} g_i \pi_0\left(g_i^{-1} v\right). \end{aligned}$$

   We will show that π is G–equivariant and that $\pi \circ \iota = \mathrm{id}_U$, so π is a splitting map. For any $g \in G$, right-multiplication by g^{-1} permutes the cosets G/H by some permutation $\sigma \in S_m$. For each i there is some $h_i \in H$ with $g^{-1}g_i = g_{\sigma(i)}h_i$. Then

$$\begin{aligned} \pi(gv) &= \frac{1}{m}\sum_{i=1}^{m} g_i \pi_0\left(g_i^{-1} g v\right) \\ &= \frac{1}{m}\sum_{i=1}^{m} g_i \pi_0\left(h_i^{-1} g_{\sigma(i)}^{-1} v\right) \\ &= \frac{1}{m}\sum_{i=1}^{m} g_i h_i^{-1} \pi_0\left(g_{\sigma(i)}^{-1} v\right) \\ &\quad \text{(since } \pi_0 \text{ is } H\text{–equivariant),} \\ &= \frac{1}{m}\sum_{i=1}^{m} g g_{\sigma(i)} \pi_0\left(g_{\sigma(i)}^{-1} v\right) \\ &= g\left(\frac{1}{m}\sum_{i=1}^{m} g_{\sigma(i)} \pi_0\left(g_{\sigma(i)}^{-1} v\right)\right) \\ &\quad \text{(since the action of } g \text{ is } \mathbb{F}_p\text{–linear),} \\ &= g\pi(v) \\ &\quad \text{(since we have simply permuted the terms of the sum defining } \pi\text{).} \end{aligned}$$

   Finally, observe that, since the inclusion ι is necessarily G–equivariant, and $\pi_0 \circ \iota = \mathrm{id}_U$,

$$\pi(\iota(u)) = \frac{1}{m}\sum_{i=1}^{m} g_i \pi_0\left(g_i^{-1}\iota(u)\right) = \frac{1}{m}\sum_{i=1}^{m} g_i \pi_0\left(\iota\left(g_i^{-1}u\right)\right) = \frac{1}{m}\sum_{i=1}^{m} g_i\left(g_i^{-1}u\right) = \frac{m}{m}u = u.$$

   Thus π is the desired splitting map, and the Splitting Lemma implies that $\ker(\pi)$ is a G–invariant direct complement of U.

4. (2 points) Define a complex $\mathbb{Z}$–representation V where $g = 1 \in \mathbb{Z}$ acts by linear map that is **not** diagonalizable. **No justification needed.**

The action of $\mathbb{Z}$ is completely specified by the image of the generator $g = 1$, which can be any invertible complex matrix. So a solution is any choice of non-diagonalizable invertible matrix. A typical example is the representation

$$\begin{aligned} \mathbb{Z} &\longrightarrow \mathrm{GL}(\mathbb{C}^2) \\ 1 &\longmapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

5. (a) (2 points) Let S and R be rings. Let N be a left S–module and M an (R, S)–bimodule. We have seen the universal property that uniquely determines the abelian group $M \otimes_S N$, and moreover that this tensor product is an R–module under the action

$$r(m \otimes n) = (rm) \otimes n \qquad \text{for } r \in R, m \in M, n \in N.$$

State a modified version of this universal property that will uniquely characterize $M \otimes_S N$ as an R–module. **No justification necessary.**
Please include definitions of any terms (such as *S–balanced*) you use.

There is a map

$$\begin{aligned} \iota : M \times N &\longrightarrow M \otimes_S N \\ (m, n) &\longmapsto m \otimes n \end{aligned}$$

The R–module $M \otimes_S N$ is uniquely determined by the following property. Given an R–module L and map

$$\phi : M \times N \to L,$$

the map ϕ factors uniquely through a map of R–modules

$$\Phi : M \otimes_S N \to L$$

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\phi"'] & M \otimes_S N \arrow[d, dashed, "\Phi"] \\ & L \end{tikzcd}$$

whenever ϕ satisfies the folloiwng properties:

- The map ϕ is *S–balanced:* For all $m, m_1, m_2 \in M, n, n_1, n_2 \in N$, and $s \in S$,

$$\begin{aligned} \phi(m_1 + m_2, n) &= \phi(m_1, n) + \phi(m_2, n), \\ \phi(m, n_1 + n_2) &= \phi(m, n_1) + \phi(m, n_2) \\ \phi(ms, n) &= \phi(m, sn). \end{aligned}$$

- The map ϕ is R–linear in the first argument. This means (along with the above $\mathbb{Z}$–linearity condition on the first argument) that

$$\phi(rm, n) = r\phi(m, n) \text{ for all } m \in M, n \in N, r \in R.$$

(b) (4 points) Let R be a ring and $S \subseteq R$ a subring. Let F be the free S–module on the set $B = \{b_1, \ldots, b_n\}$. Use the universal property of a free module and the universal property of the tensor product to give a new proof that the R–module $R \otimes_S F$ is free on the set

$$\{1 \otimes b_1, \ldots, 1 \otimes b_n\} \cong B.$$

(It is not necessary to use part (a), though you may quote part (a) without proof.)

Define maps of sets

$$\begin{aligned} B &\longrightarrow R \times F \\ b_i &\longmapsto (1, b_i) \end{aligned} \qquad\qquad \begin{aligned} B &\longrightarrow R \otimes_S F \\ b_i &\longmapsto 1 \otimes b_i \end{aligned}$$

To prove that $R \otimes_S F$ is free on B, we will show that given any R–module L and any map of sets $\varphi : B \to L$, there is a unique map of R–modules $\Phi : R \otimes_S F \to L$ making the adjacent diagram commute.

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "\varphi"'] & R \otimes_S F \arrow[d, dashed, "\Phi"] \\ & L \end{tikzcd}$$

We wish to define a map $\tilde{\varphi} : R \times F \to L$ with $\tilde{\varphi}(1, b_i) = \varphi(b_i)$ for $b_i \in B$, satisfying the criteria given in part (a). Any such map must satisfy

$$\tilde{\varphi}(1, \textstyle\sum_i s_i b_i) = \sum_i \tilde{\varphi}(1, s_i b_i) = \sum_i \tilde{\varphi}(s_i, b_i) = \sum_i s_i \tilde{\varphi}(1, b_i),$$

that is, it must be S–linear on $F \cong \{1\} \times F \subseteq R \times F$. By the universal property defining the free module F, there exists a unique S–linear map $\varphi' : F \to L$ extending φ, so let $\tilde{\varphi}(1, f) = \varphi'(f)$. To make $\tilde{\varphi}$ commute with the R–action, we must define

$$\tilde{\varphi}(r, f) = r\tilde{\varphi}(1, f) = r\varphi'(f).$$

Then for all $r, r_1, r_2 \in R$, $s \in S$, and $f, f_1, f_2 \in F$,

- $\tilde{\varphi}(r, f_1 + f_2) = r\varphi'(f_1 + f_2) = r(\varphi'(f_1) + \varphi'(f_2)) = r\varphi'(f_1) + r\varphi'(f_2) = \tilde{\varphi}(r, f_1) + \tilde{\varphi}(r, f_2)$
- $\tilde{\varphi}(rr_1 + r_2, f) = (rr_1 + r_2)\varphi'(f) = r(r_1\varphi'(f)) + r_2\varphi'(f) = r\tilde{\varphi}(r_1, f) + \tilde{\varphi}(r_2, f)$
- $\tilde{\varphi}(r, sf) = r\varphi'(sf) = rs\varphi'(f) = \tilde{\varphi}(rs, f)$

It follows from part (a) that $\tilde{\varphi}$ factors uniquely through a map $\Phi : R \otimes_S F \to L$ of R–modules making the following diagram commute.

$$\begin{tikzcd} B \arrow[r] \arrow[drrr, "\varphi"'] & F \arrow[r] \arrow[drr, dashed, "\varphi'"] & R \times F \arrow[r] \arrow[dr, dashed, "\tilde{\varphi}"] & R \otimes_S F \arrow[d, dashed, "\Phi"] \\ & & & L \end{tikzcd}$$

Since φ uniquely determines $\tilde{\varphi}$, and $\tilde{\varphi}$ uniquely determines Φ, we conclude that $R \otimes_S F$ satisfies the universal property of the free module on B.