Midterm Exam I

Math 122
2 May 2017
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Name: ____________________

Instructions: This exam has 4 questions for a total of 20 points.

The exam is closed-book. No books, notes, cell phones, calculators, or other devices are permitted. Scratch paper is available.

Fully justify your answers. You may cite any results from class or the homeworks without proof, but do give a complete statement of the result you are using.

You have 2 hours to complete the exam. If you finish early, consider checking your work for accuracy.

Jenny is available to answer questions.

Question	Points	Score
1	6	
2	3	
3	4	
4	7	
Total:	20	

1. An element e in a ring R is called *central* if $er = re$ for all $r \in R$. An element e is *idempotent* if $e^2 = e$. Let M be an R–module.

   (a) (2 points) Show that if a is central in R, then $aM = \{am \mid m \in M\}$ is a submodule of M.

   The set aM contains $a0 = 0$ and so is nonempty. Therefore by the Submodule Criterion, it suffices to check that $x + ry \in aM$ for all $x, y \in aM$ and $r \in R$. But, given arbitrary elements $am, an \in aM$ and $r \in R$, a commutes with r by assumption, so

   $$am + r(an) = am + (ra)n = am + (ar)n = am + a(rn) = a(m + rn).$$

   Since M is a submodule $(m + rn) \in M$, so we conclude that $am + r(an) \in aM$ and aM is a submodule as claimed.

   (b) (1 point) Show that if $e \in R$ is a central idempotent, then so is $(1 - e)$.

   We check that $(1 - e)$ is central: for any $r \in R$, r commutes with e, therefore

   $$r(1 - e) = r - re = r - er = (1 - e)r \qquad \text{as desired.}$$

   We check that $(1 - e)$ is idempotent: since $e^2 = e$,

   $$(1 - e)^2 = 1 - 2e + e^2 = 1 - 2e + e = 1 - e \qquad \text{as claimed.}$$

   (c) (3 points) Show that if $e \in R$ is a central idempotent, then $M \cong eM \oplus (1 - e)M$.

   Parts (a) and (b) together imply that both eM and $(1 - e)M$ are submodules of M. Thus it suffices to show that

   $$eM + (1 - e)M = M, \text{ and } eM \cap (1 - e)M = \{0\}.$$

   Given any $m \in M$, we can write

   $$m = m + em - em = em + (1 - e)m,$$

   so $m \in eM + (1 - e)M$, and we conclude $eM + (1 - e)M = M$.

   Now suppose $m \in eM \cap (1 - e)M$, so we can write $m = en = (1 - e)n'$ for some $n, n' \in M$. But multiplying through by e gives:

   $$\begin{aligned} e(en) &= e(1 - e)n' \\ e^2 n &= (e - e^2)n' \\ en &= (e - e)n' \\ en &= 0 \end{aligned}$$

   and so the $m = 0$. This proves that $eM \cap (1 - e)M = \{0\}$ and concludes the proof.

2. (3 points) Let R be a PID. Prove that any nonzero R–submodule M of R is isomorphic to R as an R–module.

Since submodules M of R are precisely the ideals of R, and R is a PID, any nonzero submodule M must be cyclically generated by a single nonzero element $a \in R$.

So suppose $M = Ra$, and consider the map

$$\begin{aligned} \phi : R &\longrightarrow M = Ra \\ r &\longmapsto ra. \end{aligned}$$

This map is an R–module homomorphism by the R–linearity Criterion: given $x, y, r \in R$,

$$\phi(x + ry) = (x + ry)a = xa + (ry)a = xa + r(ya) = \phi(x) + r\phi(y)$$

and moreover it is surjective since an arbitrary element $ra \in Ra$ has preimage $r \in R$.

Let $k \in \ker(\phi)$. Then $\phi(k) = ka = 0$. But a is nonzero and R is an integral domain, so necessarily $k = 0$.

Then by the first isomorphism theorem,

$$M = \phi(R) \cong R/\ker(\phi) = R/0 \cong R$$

and we conclude that M is isomorphic to R.

3. Let R be a commutative ring, and let M, N be R–modules.

   (a) (3 points) Suppose that M is finitely generated by $B = \{x_1, \dots, x_n\} \subset M$. Let $F \cong R^n$ be the free R–module on B. Construct an **injective** map of R–modules

   $$\mathrm{Hom}_R(M, N) \hookrightarrow \mathrm{Hom}_R(F, N).$$

   For clarity we denote the basis for F by $\{\overline{x_1}, \dots, \overline{x_n}\}$. By the universal property of the free module F, there is a uniquely defined map $\Phi : F \to M$ such that $\Phi(\overline{x_i}) = x_i$. Then Φ induces a map

   $$\begin{aligned} \mathrm{Hom}_R(M, N) &\xrightarrow{\Phi^*} \mathrm{Hom}_R(F, N) \\ f &\longmapsto f \circ \Phi \end{aligned}$$

   $$\begin{tikzcd} F \arrow[r, "\Phi"] \arrow[dr, dashed, "\Phi^*(f)"'] & M \arrow[d, "f"] \\ & N \end{tikzcd}$$

   To verify that this is map of R–modules, we use the R–linearity Criterion: for any $f, g \in \mathrm{Hom}_R(M, N)$, $r \in R$, and $x \in F$

   $$\begin{aligned} (\Phi^*(f + rg))(x) &= ((f + rg) \circ \Phi)(x) \\ &= (f + rg)(\Phi(x)) \\ &= f(\Phi(x)) + rg(\Phi(x)) \\ &= (f \circ \Phi)(x) + r(g \circ \Phi)(x) \\ &= (\Phi^*(f))(x) + r(\Phi^*(g))(x) \end{aligned}$$

   so $\Phi^*(f + rg) = \Phi^*(f) + r\Phi^*(g)$, and Φ^* is R–linear.
   We next observe that Φ is surjective, since its image contains the generating set B: given an arbitrary element $x = \sum_{i=1}^n r_i x_i \in M$,

   $$\Phi\left(\sum_{i=1}^n r_i \overline{x_i}\right) = \sum_{i=1}^n r_i \Phi(\overline{x_i}) = \sum_{i=1}^n r_i x_i \qquad \text{by the definition and } R\text{–linearity of } \Phi,$$

   so the formal sum $\sum_{i=1}^n r_i \overline{x_i} \in F$ is a preimage of x.
   We now show that Φ^* injects. If $f \in \ker(\Phi^*)$, then $f \circ \Phi$ is the zero map, so the image $\Phi(F)$ is contained in $\ker(f)$. But $\Phi(F) = M$, so we conclude that f is the zero map, and that Φ^* injects.

   (b) (1 point) Conclude that, if we additionally assume N^n is a Noetherian module, then $\mathrm{Hom}_R(M, N)$ is a finitely generated R–module.

   You proved on Homework #2 that

   $$\mathrm{Hom}_R(F, N) \cong N^n.$$

   Hence the map Φ^* realizes $\mathrm{Hom}_R(M, N)$ as an R–submodule of the R–module N^n. Since N^n is assumed to be Noetherian, this submodule must be finitely generated.

4. (a) (2 points) Classify all submodules of the $\mathbb{C}[x]$–module $V \cong \mathbb{C}^2$, where x acts by the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. **Justify your solution.**

Any submodule of V must in particular be a $\mathbb{C}$–linear subspace. The trivial submodule $\{0\}$ and the whole module V are always submodules, so it remains to determine which 1–dimensional subspaces are submodules. A subspace W is a $\mathbb{C}[x]$–submodule exactly when it is closed under the action of x, and it suffices to check that x maps a $\mathbb{C}$–basis for W back into W.

So let $W = \text{span}_{\mathbb{C}}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$ be an arbitrary 1–dimensional subspace of V. Then

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

but $\begin{bmatrix} b \\ 0 \end{bmatrix}$ is only contained in $\text{span}_{\mathbb{C}}\left(\begin{bmatrix} a \\ b \end{bmatrix}\right)$ if the y–component b is zero. We conclude that the only one-dimensional submodule of V is the x–axis, the subspace $\text{span}_{\mathbb{C}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. A complete list of $\mathbb{C}[x]$–submodules of V is:

$$\{0\}, \ \text{span}_{\mathbb{C}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right), \ V.$$

(b) (2 points) Compute the tensor product $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$. **Justify your solution.**

We will prove $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = \{0\}$.
(In short, this is because $\mathbb{Q}/\mathbb{Z}$ is both *divisible* and *torsion*.)
Since the tensor product is generated by simple tensors $m \otimes n$, it suffices to show that these vanish. So suppose that m is the equivalence class of $\frac{a}{b}$ and that n is the equivalence class of $\frac{c}{d}$ in $\mathbb{Q}/\mathbb{Z}$, with $a, b, c, d \in \mathbb{Z}$. Then

$$\begin{aligned}
\frac{a}{b} \pmod{\mathbb{Z}} \otimes \frac{c}{d} \pmod{\mathbb{Z}} &= \frac{a}{b} \pmod{\mathbb{Z}} \otimes \frac{bc}{bd} \pmod{\mathbb{Z}} \\
&= \frac{a}{b} \pmod{\mathbb{Z}} \otimes b\left(\frac{c}{bd} \pmod{\mathbb{Z}}\right) \\
&= \left(\frac{a}{b} \pmod{\mathbb{Z}}\right) b \otimes \frac{c}{bd} \pmod{\mathbb{Z}} \\
&= \frac{ab}{b} \pmod{\mathbb{Z}} \otimes \frac{c}{bd} \pmod{\mathbb{Z}} \\
&= 0 \pmod{\mathbb{Z}} \otimes \frac{c}{bd} \pmod{\mathbb{Z}} \\
&= 0.
\end{aligned}$$

(c) (3 points) When we apply the contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/3\mathbb{Z})$ to the short exact sequence of abelian groups

$$\begin{array}{c} 0 \longrightarrow \mathbb{Z}/3\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/9\mathbb{Z} \xrightarrow{\phi} \mathbb{Z}/3\mathbb{Z} \longrightarrow 0 \\ a \longmapsto 3a, \;\; b \longmapsto b \bmod 3 \end{array}$$

then we obtain a sequence of abelian groups

$$0 \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xleftarrow{\psi^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xleftarrow{\phi^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \longleftarrow 0$$

Compute this sequence by identifying each abelian group $\mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Z}/3\mathbb{Z})$ as a (productive of) cyclic group(s), and explicitly describing where ψ^* and ϕ^* map a set of generators of these abelian groups. **Justify your solution.**

You proved on Homework #1 that there are isomorphisms of abelian groups

$$\begin{array}{c} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/3\mathbb{Z} \\ f \longmapsto f(1) \end{array} \qquad \begin{array}{c} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}/3\mathbb{Z} \\ g \longmapsto g(1) \end{array}$$

The group $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$ is cyclically generated by the map f_1 such that $f_1(1) = 1 \in \mathbb{Z}/3\mathbb{Z}$, and the group $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$ is cyclically generated by the map g_1 such that $g_1(1) = 1 \in \mathbb{Z}/3\mathbb{Z}$.

To compute the map ϕ^*, we must determine where it sends f_1.

We will identify the resulting morphism

$$\phi^*(f_1) : \mathbb{Z}/9\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$$

based on where it maps $1 \in \mathbb{Z}/9\mathbb{Z}$.

$$\begin{tikzcd} \mathbb{Z}/9\mathbb{Z} \arrow[r, "\text{mod } 3", "\phi"'] \arrow[dr, dashed, "\phi^*(f_1)"'] & \mathbb{Z}/3\mathbb{Z} \arrow[d, "f_1"] \\ & \mathbb{Z}/3\mathbb{Z} \end{tikzcd}$$

By definition,

$$\phi^*(f_1)(1) = f_1(\phi(1)) = f_1(1) = 1,$$

so ϕ^* maps f_1 to the generator g_1 of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$.

Next we compute the map $\psi^*(g_1)$. We see

$$\psi^*(g_1)(1) = g_1(\psi(1)) = g_1(3) = 3 \equiv 0 \in \mathbb{Z}/3\mathbb{Z},$$

so $\psi^*(g_1)$ is the zero map on $\mathbb{Z}/3\mathbb{Z}$, and ψ^* is the zero map on $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$.

$$\begin{tikzcd} \mathbb{Z}/3\mathbb{Z} \arrow[r, "3", "\psi"'] \arrow[dr, dashed, "\psi^*(g_1)"'] & \mathbb{Z}/9\mathbb{Z} \arrow[d, "g_1"] \\ & \mathbb{Z}/3\mathbb{Z} \end{tikzcd}$$

The resulting sequence is

$$\begin{array}{c} 0 \longleftarrow \mathbb{Z}/3\mathbb{Z} \xleftarrow{\psi^*=0} \mathbb{Z}/3\mathbb{Z} \xleftarrow{\phi^*=\mathrm{id}} \mathbb{Z}/3\mathbb{Z} \longleftarrow 0 \\ 0 \;\longleftarrow\!\shortmid\; b, \;\; a \;\longleftarrow\!\shortmid\; a \end{array}$$