1. Recall that a *group action* of a group G on a set S is a map $G \times S \to S$ such that

$$g \cdot (h \cdot a) = (gh) \cdot a \text{ for all } g, h \in G, a \in S \qquad \text{and} \qquad 1 \cdot a = a \text{ for all } a \in S.$$

   Let M be an R–module. Show that this structure defines a group action of the multiplicative group $R^\times$ on the set M.

2. Let $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ be an ascending chain of nested submodules of an R–module M. Show that $\bigcup_i M_i$ is a submodule of M.

3. Classify all submodules of the $\mathbb{C}[x]$–module ...
   (a) $\mathbb{C}^2$ with an action of x by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
   (b) $\mathbb{C}[x]/(x-\pi)^2$.
   (c) $\mathbb{C}^3$ where x acts by a diagonalizable matrix with eigenvalues $1, 1, 2$.

4. Classify the subgroups of the abelian group $\mathbb{Q}/\mathbb{Z}$.

5. Let R and S be commutative rings. Prove or disprove: For any ideal I in the product ring $R \times S$, I can be expressed as a product $I_R \times I_S$ where I_R is an ideal in R, and I_S an ideal in S.

6. Let M be an R–module.
   (a) Prove that its annihilator $\text{ann}(M)$ is a two-sided ideal of R, and that there is a well-defined action of $R/\text{ann}(M)$ on M.
   (b) Prove that this action is faithful.

7. Let M be an R–module, I a (right) ideal of R, and N a R–submodule. Prove the following statement, or find a counterexample:

$$\text{If } \text{ann}(I) = N, \text{ then } \text{ann}(N) = I.$$

   If the statement is false, determine whether you can replace the second equality $=$ with either $\subseteq$ or $\supseteq$ to make a true statement.

8. Suppose that M, N are modules over a ring R, and that $S \subset R$ is a subring. Recall that M and N inherit S–module structures.
   (a) Show that $\text{Hom}_R(M, N) \subseteq \text{Hom}_S(M, N)$.
   (b) Find an example of a ring R, a proper subring $S \subset R$, and nonzero R–modules M and N so that $\text{Hom}_R(M, N) = \text{Hom}_S(M, N)$.
   (c) Find an example of a rings $S \subseteq R$, and R–modules M and N so that $\text{Hom}_R(M, N) \neq \text{Hom}_S(M, N)$.

9. Let A, B, C, X be R–modules. Prove or disprove: if $A = B \oplus C$, then $(A \cap X) = (B \cap X) \oplus (C \cap X)$.

10. Let M be an R–module.
    (a) Define $\text{Tor}(M)$.
    (b) Show by example that if R is not commutative, then $\text{Tor}(M)$ may not be a submodule of M.

11. Let R be an integral domain, and M an R–module. Prove that $M/\text{Tor}(M)$ is torision-free.

12. (a) Let $\mathbb{F}$ be a field. Let V be an $\mathbb{F}[x]$ module where x acts by a linear transformation T. Under what conditions will an $\mathbb{F}$–linear map $S : V \to V$ be an element of $\text{End}_{\mathbb{F}[x]}(V)$?
    (b) Let $R = \mathbb{Z}[\sqrt{2}] = \{a + \sqrt{2} \mid a, b \in \mathbb{Z}\}$ and consider R as a module over itself. As an abelian group, $M = \mathbb{Z}[\sqrt{2}]$ is isomorphic to $\mathbb{Z}^2$ under the identification $a + b\sqrt{2}$ with $(a, b) \in \mathbb{Z}^2$. Show that $\text{End}_{\mathbb{Z}}(M)$ is the matrix ring $M_{2\times 2}(\mathbb{Z})$, and identify the subset of matrices $\text{End}_R(M) \subseteq M_{2\times 2}(\mathbb{Z})$ that commute with the action of R.

13. State and prove the first isomorphism theorem for R–modules.

14. Give an example of an integral domain R and two non-isomorphic finitely generated torsion R–modules with the same annihilators.

15. Let R be a commutative ring.
    (a) Suppose R has the property that every ideal is principal. Show that every R–submodule of the module R is cyclic.
    (b) Show additionally that if N is quotient of the R–module R, then every submodule of N is cyclic.
    (c) Let R be PID and $R \to S$ a surjective ring map. Show that every ideal of S is principal. Show by example that S need not be an integral domain.

16. Let V be a $\mathbb{C}[x]$–module such that V is finite dimensional as a vector space over $\mathbb{C}$. Prove that V is a torsion module.

17. Let $\phi : M \to N$ be a homomorphism of R–modules. Let I be a right ideal of R. Let $\text{ann}_M(I)$ denote the annihilator of I in M, and $\text{ann}_N(I)$ the annihilator of I in N. Prove or find a counterexample: $\phi(\text{ann}_M(I)) \subseteq \text{ann}_N(I)$.

18. Let M and N be R–modules, and I an ideal of R contained in $\text{ann}(M)$ and $\text{ann}(N)$. Show that any map of R–modules $\phi : M \to N$ is also a map of (R/I)–modules. Conclude that $\text{Hom}_R(M, N) = \text{Hom}_{R/I}(M, N)$.

19. Let R be a ring and $S \subseteq R$ a subring. An R–module M has the structure of an S–module under restriction of scalars. Show that if M is finitely generated as an S–module, then it is finitely generated as an R–module. Is the converse true?

20. (a) If $a \in R$, prove that $Ra \cong R/\text{ann}(a)$, where $\text{ann}(a)$ denotes the annihilator of the left ideal generated by a.
    (b) Let M be an R–module. For $a, b \in M$, let $A = \{a, b\}$. Prove or disprove: $RA \cong R/I$, where I is the annihilator of the submodule generated by a and b.

21. Let M be and R–module with submodules A and B. Prove that the map $A \times B \longrightarrow A + B$ is an isomorphism if and only if $A \cap B = \{0\}$.

22. Give an example of a finitely generated R–module M and a submodule that is not finitely generated.

23. (a) Let R be an integral domain. An R–module M is *torsion* if $M = \text{Tor}(M)$, that is, for every $m \in M$ there is some nonzero $r \in R$ so that $rm = 0$. Let A be a finitely generated R–module and B a submodule. Show that A/B is torsion if and only if there is some nonzero $r \in R$ so that $rA \subseteq B$.
    (b) Show by example that this result can fail without the assumption that A is finitely generated. (Hint: Consider the $\mathbb{Z}$–module $\mathbb{Q}$).

24. Let S be the set of all sequences of integers $(a_1, a_2, a_3, \ldots)$ that are nonzero in only finitely many components (in other words, all functions $\mathbb{N} \to \mathbb{Z}$ with finite support). Verify that S is a ring (without identity) under componentwise addition and multiplication. Is S a finitely generated S–module?

25. Let R be a ring.
    (a) Give the definition of a *free* R–module on a set A.
    (b) Given a set A, explain how to construct a free R–module $F(A)$ on A.
    (c) State the universal property for a free R–module.
    (d) Verify that $F(A)$ satisfies this universal property.
    (e) Prove that the universal property determines $F(A)$ uniquely up to unique isomorphism.
    (f) Show that F defines a covariant functor from the category of sets to the category of R–modules.

26. Let M be an R–module, and $x_1, x_2, \ldots, x_n \in M$.
    (a) Define what it means for $x_1, x_2, \ldots, x_n$ to be *R–linearly independent*. Show that $x_1, x_2, \ldots, x_n$ are R–linearly independent if and only if the following map is injective:

$$R^n = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n \longrightarrow M$$
$$e_i \longmapsto x_i$$

    (b) Define what it means for $B = \{x_1, x_2, \ldots, x_n\}$ to be a *basis* for RB. Show that B is a basis for RB if and only if $x_1, \ldots, x_n$ are linearly independent.

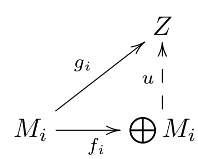
27. Let R be a commutative ring. Show that every R–module is free if and only if R is a field.

28. Suppose that R is a ring and that S is a subring.
    (a) Suppose that F is a free R–module. Prove or disprove: F is a free S–module after restriction of scalars to S.
    (b) Suppose that M is an R–module that is free as an S–module after restriction to S. Prove or disprove: M must be a free R–module.

29. Let R be a commutative ring. If M and N are free R–modules, will the R–module $\mathrm{Hom}_R(M, N)$ be free? If $\mathrm{Hom}_R(M, N)$ is a free R–module, must M and N be free?

30. Let R be a commutative ring, and let A, B, M be R–modules. Prove isomorphisms of R–modules:
    (a) $\mathrm{Hom}_R(A \oplus B, M) \cong \mathrm{Hom}_R(A, M) \oplus \mathrm{Hom}_R(B, M)$
    (b) $\mathrm{Hom}_R(M, A \oplus B) \cong \mathrm{Hom}_R(M, A) \oplus \mathrm{Hom}_R(M, B)$

31. **(Coproducts of families).** Prove that the direct sum of R–modules $\bigoplus_{i\in I} M_i$, along with the inclusions $f_i : M_i \to \bigoplus_{i\in I} M_i$, satisfies the following universal property: whenever there is a family of maps $\{g_i : M_i \to Z \mid i \in I\}$ there is a unique map u making the following diagrams commute:



    Explain why this universal property can be taken as the definition of the direct sum of R–modules.

32. State the definition of a category, and the definition of a covariant functor.

33. Let $\mathcal{C}$ be a category. Prove that if $X \in \mathrm{ob}(\mathcal{C})$, then the identity morphism id_X is unique.

34. Prove that if $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an isomorphism, then its inverse f^{-1} is unique.

35. (a) Prove that in the category of sets, a map is monic iff it is injective, and epic iff it is surjective.
    (b) Prove that in any category the composition of monomorphisms (respectively, epimorphisms, or isomorphisms) is a monomorphisms (respectively, an epimorphism, or isomorphism).
    (c) Prove that isomorphisms are both monic and epic.

36. Prove or disprove the following statements.
    (a) If $f : A \to B$ is a monomorphism (respectively, epimorphism) in a category $\mathcal{C}$, then the image of f under any (covariant) functor $\mathcal{C} \to \mathcal{D}$ must be a monomorphism (respectively, epimorphism) in $\mathcal{D}$.
    (b) If $0 \to A \to B \to C \to 0$ is a short exact sequence in the category of R–modules, then its image under any (covariant) functor R–<u>Mod</u> $\to R$–<u>Mod</u> must be an exact sequence.

37. Let $\underline{\text{Grp}}$ be the category of groups and group homomorphisms. Let Z be the map $Z : \underline{\text{Grp}} \to \underline{\text{Grp}}$ that maps a group G to its centre $Z(G) = \{a \in G \mid ag = ga \ \ \forall g \in G\}$. Show that Z **cannot** be made into a functor by defining it to take a map of groups $f : G \to H$ to the restriction $f|_{Z(G)}$ of f to $Z(G)$, since $f(Z(G))$ may not be contained in $Z(H)$.

38. Let R be a ring. Consider the map on the objects of R–$\underline{\text{Mod}}$ that takes and R–module M to the submodule $\text{ann}(R)$, and takes a morphism of R–modules $f : M \to N$ to its restriction $f|_{\text{ann}(R)}$ to the submodule $\text{ann}(R) \subseteq M$. Does this give a well-defined functor R–$\underline{\text{Mod}} \to R$–$\underline{\text{Mod}}$?

39. Let R be an integral domain. Prove or disprove: The map of R–modules that takes an R–module M to its R–submodule $\text{Tor}(M)$ and takes a map $f : M \to N$ to its restriction $f|_{\text{Tor}(M)}$ defines a covariant functor R–$\underline{\text{Mod}} \to R$–$\underline{\text{Mod}}$.

40. Let R be a ring. Define a functor on the category R–$\underline{\text{Mod}}$ that takes an R–module M to the R–module $M \oplus M$. Verify that your construction is functorial.

41. An object I in a category $\mathcal{C}$ is called *initial* if for every object X in $\mathcal{C}$ there is a unique morphism $I \to X$ in $\mathcal{C}$. An object T is called terminal if for every object X in $\mathcal{C}$ there is a unique morphism $X \to T$.
    (a) Show that if $\mathcal{C}$ has an initial object I, then this object is unique up to unique isomorphism. Similarly show that terminal objects are unique up to unique isomorphism.
    (b) Show that the empty set is initial in $\underline{\text{Set}}$, and that the singleton set is terminal in $\underline{\text{Set}}$.
    (c) Find an example of a category with no initial or terminal object.
    (d) Show that in R–$\underline{\text{Mod}}$, the zero module is both initial and terminal.
    (e) Identify an initial object and a terminal object in the category of rings (with unit) and (unit-preserving) ring maps.

42. Let R be a ring and M a fixed R–module. Verify that the following maps each define a functor of categories. Explain how to define the functor on morphisms, determine whether it is covariant or contravariant, and verify that the map is functorial.

    (a)
    $$\begin{aligned} \text{Hom}_R(M,-) : R\text{–}\underline{\text{Mod}} &\longrightarrow \underline{\text{Ab}} \\ N &\longmapsto \text{Hom}_R(M,N) \end{aligned}$$

    (b)
    $$\begin{aligned} \text{Hom}_R(-,M) : R\text{–}\underline{\text{Mod}} &\longrightarrow \underline{\text{Ab}} \\ N &\longmapsto \text{Hom}_R(N,M) \end{aligned}$$

43. There is a single nonzero map of abelian groups $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, and a single nonzero map $g : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Compute the images of these maps under the functors

    (a) $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, -)$ (b) $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, -)$ (c) $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2\mathbb{Z})$ (d) $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/4\mathbb{Z})$

    This means, express the image of each abelian group as a cyclic abelian group, and determine how the induced maps f_*, f^*, g_*, g^* act on the generator of these cyclic groups.

44. Find two non-isomorphic extensions of the abelian groups $\mathbb{Z}/2\mathbb{Z}$ by $\mathbb{Z}/6\mathbb{Z}$.

45. The rows of the following diagram are exact. Prove that if m and p are surjective and q is a injective, then n is surjective.

$$\begin{array}{ccccccc} B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{j} & E \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} \\ B' & \xrightarrow[s]{} & C' & \xrightarrow[t]{} & D' & \xrightarrow[u]{} & E' \end{array}$$

46. An R–submodule N of an R-module M has a *direct complement* P if $M \cong N \oplus P$ for P an R–submodule.
   (a) Show that the $\mathbb{Z}$–submodule $2\mathbb{Z} \subseteq \mathbb{Z}$ does not have a direct complement.
   (b) Show that the $\mathbb{Z}$–submodule $(3) \subseteq \mathbb{Z}/9\mathbb{Z}$ does not have a direct complement.
   (c) Show that the $\mathbb{Z}$–submodule $(3) \subseteq \mathbb{Z}/6\mathbb{Z}$ *does* have a direct complement.
   (d) Let V be the the $\mathbb{Q}[x]$–module where x acts by the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Show that $U = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ is a submodule of V with no direct complement.
   (e) Show that the $\mathbb{Q}[x]$–submodule $(x) \subseteq \mathbb{Q}[x]/(x^3)$ does not have a direct complement.
   (f) Show that every linear subspace of a vector space has a direct complement.

47. Re-interpret each of the parts of Problem (46) in terms of the splitting or non-splitting of a short exact sequence.

48. Let $0 \to A \to B \to C \to 0$ be an exact sequence of R–modules.
   (a) Show that if B is torsion, so are A and C.
   (b) Prove or find a counterexample: If A and C are torsion, then so is B.
   (c) Show by example that if B is torsion-free, then A is torsion-free, but C need not be.
   (d) Prove or find a counterexample: If A and C are torsion-free, then so is B.

49. Let M be a right R–module, and N a left R–module.
   (a) Describe an explicit construction of the tensor product $M \otimes_R N$ as a quotient of abelian groups.
   (b) State the universal property of the tensor product.
   (c) Verify that the explicit construction satisfies the universal property.

50. Compute the following tensor products of abelian groups:

$$(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} \qquad (\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \qquad (\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \qquad (\mathbb{Q}/\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

$$(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \ (m, n \text{ coprime}) \qquad (\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n^2\mathbb{Z})$$