Reading: Dummit–Foote Ch 10.1.

Please review the Math 122 Course Information posted on our webpage:
`http://web.stanford.edu/~jchw/2018Math122`.

## Summary of definitions and main results

**Definitions we've covered:** left $R$–module, right $R$–module, $R$–submodule, endomorphism, free $R$–module of rank $n$, annihilator of a submodule, annihilator of a (right) ideal.

**Main results:** Two equivalent definitions of an $R$–module; the submodule criterion, equivalence of vector spaces over a field $\mathbb{F}$ and $\mathbb{F}$–modules; equivalence of abelian groups and $\mathbb{Z}$–modules; if $I$ annihilates an $R$–module $M$ then $M$ inherits a $(R/I)$–module structure; structure of an $\mathbb{F}[x]$–module for a field $\mathbb{F}$.

## Warm-Up Questions

The "warm-up" questions do not need to be submitted (and won't be graded), however, you're encouraged to work out their solutions!

1. State the definition / axioms for a ring $R$ (which we assume has unit 1).
2. In class we gave the definition of a left $R$–module. Formulate the definition of a *right $R$–module $M$*.
3. Let $R$ be a ring with 1 and $M$ a left $R$–module. Prove the following:
   (a) $0m = 0$ for all $m$ in $M$.
   (b) $(-1)m = -m$ for all $m$ in $M$.
   (c) If $r \in R$ has a left inverse, and $m \in M$, then $rm = 0$ only if $m = 0$.
4. Show that if $R$ is a commutative ring, then a left $R$–module structure on an abelian group $M$ also defines a right $R$–module on $M$ and vice versa. Is this true for noncommutative rings $R$?
5. (**Restriction of scalars**). Let $M$ be an $R$–module, and let $S$ be any subring of $R$. Explain how the $R$–module structure on $M$ also gives $M$ the structure of an $S$–module. This operation is called *restriction of scalars* from $R$ to the subring $S$.
6. Verify that the axioms for a vector space over a field $\mathbb{F}$ are equivalent to the axioms for an $\mathbb{F}$–module.
7. Verify that the axioms for an abelian group $M$ are equivalent to the axioms for a $\mathbb{Z}$–module structure on $M$. How does an integer $n$ act on $m \in M$?
8. Let $\mathbb{F}$ be a field, and $x$ a formal variable. Prove that modules $V$ over the polynomial ring $\mathbb{F}[x]$ are precisely $\mathbb{F}$–vector spaces $V$ with a choice of linear map $T : V \to V$. In Assignment Problem 4 we will see that different maps $T$ give different $\mathbb{F}[x]$–module structures on $V$.
9. Prove the *submodule criterion*: If $M$ is a left $R$–module and $N$ a subset of $M$, then $N$ is a left $R$–submodule if and only if
$$(i) \quad N \neq \varnothing \qquad \text{and} \qquad (ii) \quad x + ry \in N \text{ for all } x, y \in N \text{ and all } r \in R.$$
10. Consider $R$ as a module over itself. Prove that the $R$–submodules of the module $R$ are precisely the left ideals $I$ of $R$.
11. Let $R^n$ be the free module of rank $n$ over $R$. Prove that the following are submodules:
    (a) $I_1 \times I_2 \times \cdots \times I_n$, with $I_i$ a left ideal of $R$.

(b) The $i^{th}$ direct summand $R$ of $R^n$.

(c) $\{(a_1, a_2, \ldots, a_n) \in R^n \mid a_1 + a_2 + \cdots + a_n = 0\}$.

12. Let $M$ be a left $R$–module. Show that the intersection of a (nonempty) collection of submodules is a submodule.

13. (a) Let $M$ be an $R$–module and $N$ an $R$–submodule. Prove that the annihilator $\mathrm{ann}(N)$ is a 2-sided ideal of $R$.

    (b) Let $M$ be an $R$–module and $I$ a right ideal of $R$. Show that $\mathrm{ann}(I)$ is an $R$–submodule of $M$.

    (c) Compute the annihilator of the ideal $3\mathbb{Z} \subseteq \mathbb{Z}$ in the $\mathbb{Z}$–module $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/15\mathbb{Z}$.

14. (a) For $p$ prime, an *elementary abelian p-group* is an abelian group $G$ where $pg = 0$ for all $g \in G$. Prove that an elementary abelian $p$-group is a $\mathbb{Z}/p\mathbb{Z}$–module, equivalently, an $\mathbb{F}_p$–vector space.

    (b) Conversely, show that any $\mathbb{Z}/p\mathbb{Z}$–module $M$ must satisfy $pm = 0$ for all $m \in M$, in other words, the underlying abelian group $M$ must be an elementary abelian $p$-group.

15. Let $M$ be a $\mathbb{Z}$–module. Fix an integer $n > 1$. Under what conditions on $M$ does the action of $\mathbb{Z}$ on $M$ induce an action of $\mathbb{Z}/n\mathbb{Z}$ on $M$?

16. A student makes the following claim: "Since $\mathbb{Z}/2\mathbb{Z}$ is a subring of $\mathbb{Z}/4\mathbb{Z}$, we can let $\mathbb{Z}/2\mathbb{Z}$ act by left multiplication to give $\mathbb{Z}/4\mathbb{Z}$ the structure of a $\mathbb{Z}/2\mathbb{Z}$–module. Then $\mathbb{Z}/4\mathbb{Z}$ is a $\mathbb{Z}/2\mathbb{Z}$–vector space with 4 elements, so it must be isomorphic as a vector space to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$." Prove that $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ are not even isomorphic as abelian groups, and find the flaw in this argument.

17. Let $M$ be an $R$–module, and consider $\mathrm{Tor}(M)$ as defined in Assignment Question 2.

    (a) Find $\mathrm{Tor}(\mathbb{Z}/7\mathbb{Z})$ if $\mathbb{Z}/7\mathbb{Z}$ is consider a module over (i) $\mathbb{Z}$, (ii) $\mathbb{Z}/7\mathbb{Z}$, or (iii) $\mathbb{Z}/21\mathbb{Z}$.

    (b) Show that if $R$ has zero divisors, then every nonzero $R$–module has nonzero torsion elements.

18. **(Group theory review)** State the structure theorem for finitely generated abelian groups.

19. **(Linear algebra review)**

    (a) Define the following terms (as they apply to finite dimensional vector spaces)

    - *vector space* over $\mathbb{F}$; *vector subspace*
    - *linear dependence* and *linear independence* of a set of vectors
    - *spanning set* of vectors for a vector subspace
    - *basis* and *dimension* of a vector subspace
    - the *direct sum* of vector subspaces

    (b) If you have not already seen proofs that

    - linearly independent sets of vectors in a finite dimensional vector space $V$ can be extended to a basis, and
    - all bases for $V$ have the same cardinality so $\dim(V)$ is well-defined

    then take a look at Dummit-Foote Chapter 11.1.

    (c) Let $T$ be a linear transformation on a finite-dimensional $\mathbb{F}$-vector space $V$. Define an *eigenvector* of $T$ and its associated *eigenvalue*. Find all eigenvectors and eigenvalues of the following matrices, over $\mathbb{R}$ and over $\mathbb{C}$.

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

    (d) If $T$ has a basis of eigenvectors, then such as basis is called an *eigenbasis*. What can you say about the structure of a matrix with an eigenbasis, and why is this important? Which of the above four matrices have eigenbases over $\mathbb{R}$, or over $\mathbb{C}$?

# Assignment Questions

The following questions should be handed in. Fully justify your solutions.

1. Let $M$ be an abelian group (with addition), and $R$ a ring.
   (a) Define an *endomorphism* of $M$, and show that the set of endomorphisms $\mathrm{End}(M)$ of $M$ form a ring under composition and pointwise addition.
   (b) Prove that a left $R$–module structure on $M$ is equivalent to the data of a homomorphisms of rings $R \to \mathrm{End}(M)$. Use this result to formulate an alternative definition of a left $R$–module.
   (c) What should the analogous definition be for right $R$–modules?
   (d) We have another name for the kernel of the map $R \to \mathrm{End}(M)$. What is it?
   (e) Let $M$ be an $R$–module, and $\phi : S \to R$ a homomorphism of rings. Show how the map $\phi$ can be used to define an $S$–module structure on $M$. Explain why *restriction of scalars* is a special case of this construction. (Warm up Problem 5.)

2. An element $m$ in an $R$–module $M$ is called a *torsion element* if $rm = 0$ for some nonzero $r \in R$. The set of torsion elements is denoted
$$\mathrm{Tor}(M) := \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$
   Prove that if $R$ is an integral domain, then $\mathrm{Tor}(M)$ is submodule of $M$.
   *Remark:* For commutative rings $R$, some sources only define $\mathrm{Tor}(M)$ with respect to elements $r \in R$ that are not zero divisors.

3. For each of the following statements, prove that equality holds or find a counterexample. If equality does not hold in the conclusion of the statement, then determine whether you can prove containment of sets in one direction. Let $M$ be an $R$–module, $I$ a (right) ideal of $R$, and $N$ a $R$–submodule.
   (a) If $\mathrm{ann}(N) = I$, then $\mathrm{ann}(I) = N$.
   (b) If $\mathrm{ann}(I) = N$, then $\mathrm{ann}(N) = I$.

4. Let $\mathbb{F}$ be a field. Let $V$ be a module over the polynomial ring $\mathbb{F}[x]$. For each of the following, classify all submodules of $V$.
   (a) $V = \mathbb{F}^2$, and $x$ acts by scalar multiplication by 2.
   (b) $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^2$, and $x$ acts by rotation by $\frac{\pi}{2}$.
   (c) $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^2$, and $x$ acts by a matrix which has a basis of eigenvectors $v_1$ and $v_2$ with distinct eigenvalues $\lambda_1$ and $\lambda_2$, respectively.
   (d) $V = \mathbb{F}[x]$, and $x$ acts by multiplication by $x$ as usual.
   (e) $V = \mathbb{F}[x]/(x^2)$, and $x$ acts by multiplication by $x$ as usual.